Course Note – March 16, 2005
IEOR 251 – Facility Design and Logistics
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The 1- Tree Lower Bound for TSP

1-Tree
Definition: For a given vertex, say vertex 1, a 1-Tree is a tree of {2,3,…n} +2 distinct edges connected to vertex 1.
1-Tree has precisely one cycle.

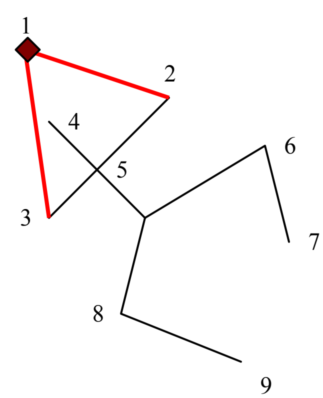


Fig. 1 Example of 1-Tree

Minimum Weight 1-Tree: Min cost 1-tree of all possible 1-Trees.
To find minimum Weight 1- Tree, First Find minimum spanning tree of {2,3,…n} vertices, and add two lowest cost edges incident to vertex 1.

Any TSP Tour is 1- Tree tour (with arbitrary starting node 1) in which each vertex has a degree of 2. If Minimum weight 1- Tree is a tour, it is the optimal TSP tour. Thus, the minimum 1- Tree provides a lower bound on the length of the optimal TSP tour.

Improving the 1- Tree Lower Bound

Consider Vector $\pi_i = \{\pi_1, \pi_2, \ldots, \pi_n\}$,
Distance Matrix $\{d_{ij}\}$
Now, transform the distances as follows
$d_{ij}' = \pi_i + \pi_j + d_{ij}$

And Note that if L is the length of nay tour then each node appears twice, so the length of the tour with new distances is

$$L + 2\sum_i \pi_i$$

But, Min 1-Tree does change.

Enumerate all possible 1-trees. Let d_i^k be the degree of the i^{th} node in the k^{th} tree, T_k be the cost of k^{th} tree using original distances.
The cost of k^{th} tree in transformed distance matrix is

$$T_k + \sum_{i \in V} d_i^k \pi_i$$

Thus, the minimum weight 1-tree on the transformed distance matrix

$$L(\pi) = \min_k \{T_k + \sum_{i \in V} d_i^k \pi_i\}$$

$$L^* + 2\sum_i \pi_i \geq \min_k \{T_k + \sum_{i \in V} d_i^k \pi_i\}$$

$$L^* \geq \min_k \{T_k + \sum_{i \in V} (d_i^k - 2)\pi_i\} = w(\pi)$$

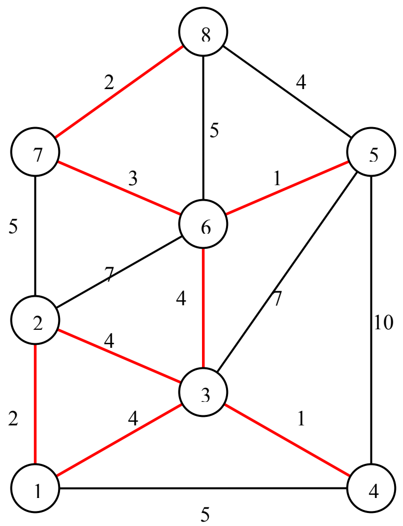
So, the best Lower Bound comes from maximizing $w(\pi)$ over all values of π.
For a given π, problem is easy to solve.
So we need to find best π's.

Held and Kalp (1970,1971) use subgradient method.

$$\pi_i^{j+1} = \pi_i^j + t_j(d_i^j - 2)$$

$$t_j = \frac{\lambda_j (UB - w(\pi^j))}{\sum_{i=1}^{n} (d_i^j - 2)^2}$$

Example

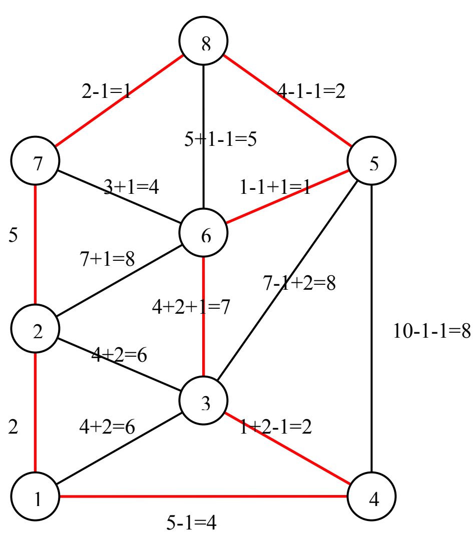


UB=25 , π^0=0, L(π^0)=21

d_i^0={2,2,4,1,1,3,2,1} – initial degree of node for minimum weight 1- tree.

$$t_0 = \frac{2(25-21)}{4+1+1+1+1} = 1$$

$$\pi^1 = \pi^0 + t_0(d_i^0 - 2) = (0,0,2,-1,-1,1,0,-1)$$



Relating 1-Tree Lower Bound and Lagrangian Relaxation

Edges $e \subset E$
d_e cost of edge e

$$X_e = \begin{cases} 1 & if \quad e \ \ in \ \ tour \\ 0 & o/w \end{cases}$$

Given a subset $S \subset V$
Let E(S) set of all edges from E with both end points in S
Let δ(S) set of all edges in cut separating S from V\S

TSP can be formulated as follows

$$P': \ Z^* = \min \sum_{e \in E} d_e X_e$$

$$\text{s.t} \quad \sum_{e \in \delta(i)} X_e = 2, \forall i = 1,2,...n \qquad (1)$$

$$\sum_{e \in E(S)} X_e \le |S| - 1, \forall S \subseteq V \setminus \{1\}, S \ne 0$$

$$0 \le X_e \le 1, X_e : Integer$$

Claim that constraint (1) can be replaced by the following constraints

$$\sum_{e \in \delta(i)} X_e = 2, \forall i = 1,2,...n-1 \qquad (2)$$

$$\sum_{e \in E} X_e = n$$

Proof>
This is true for i=1,…n-1, so must prove for i="n" constraint.

$$\sum_{e \in E} X_e = \frac{1}{2} \sum_{i=1}^{n} \sum_{e \in \delta(i)} X_e$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{e \in \delta(i)} X_e + \frac{1}{2} \sum_{e \in \delta(n)} X_e$$

$$= (n-1) + \frac{1}{2} \sum_{e \in \delta(n)} X_e$$

Thus, $\sum_{e \in E} X_e = n$ if and only if $\sum_{e \in \delta(n)} X_e = 2$ ■

Relaxing Constraint (2),

$$\max_u \{\min_v \sum_{e\in E} d_e X_e + \sum_{i=1}^{n-1} u_i (\sum_{e\in S(i)} X_e - 2)\}$$

$$\text{s.t} \quad \sum_{e\in E} X_e = n$$

$$\sum_{e\in E(S)} X_e \le |S| - 1, \forall S \subseteq V \setminus \{1\}, S \ne 0$$

$$X_e : Binary$$

But, these constraints are met by all 1-trees, Edmonds in 1971 showed that the extreme points of this is the set of all 1-trees.

Theorem: Wolsey(1980)

Z^* : optimal TSP Tour

$Z^{\text{1-tree}}$:1-tree Lower Bound

$Z^* \le \frac{3}{2} Z^{\text{1-tree}}$