Course Notes – February 16, 2005
IEOR 251 – Facility Design and Logistics
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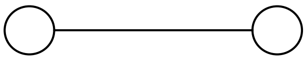
CHRISTOFIDES' HEURISTIC

Currently, best worst-case bound for triangle inequality T.S.P.

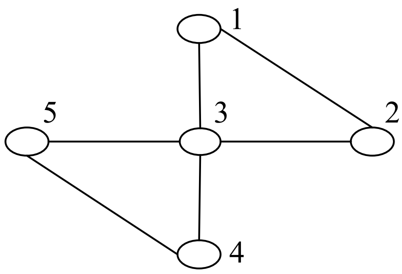
Lemma

Given a connected graph with at least two vertices, the number of vertices with odd degree is even.



Definition

An *Eulerian Tour* is a tour that traverses all edges of a graph exactly once.



Eulerian Tour: 1->2->3->4->5->3->1

Lemma

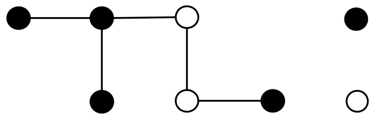
A connected graph is Eularian if and only if the degree of each vertex is even.

Definition

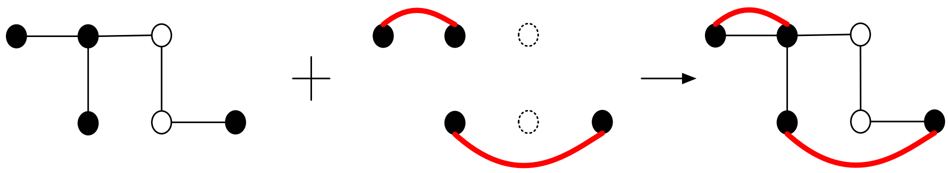
Given a graph with an even number of nodes, a *matching* is a subset of arcs such that each node is an endpoint of exactly one edge of subset.

Heuristic

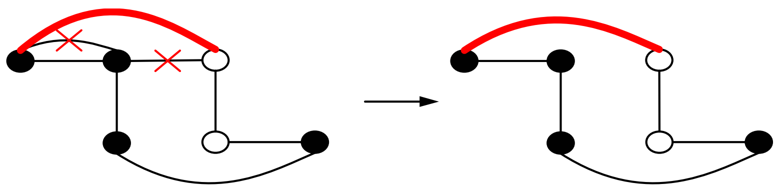
Step1. Start with minimum spanning tree – some nodes have odd degree.



Step2. Find the minimum cost matching on odd degree nodes. Adding matching edges makes the degree of all nodes even. This creates an Eulerian Graph.



Step3. Now, apply shortcut to get TSP tour.



L^C is the length of the tour generated by this heuristic.

Theorem

$$\frac{L^C}{L^*} \leq \frac{3}{2}$$

Proof:

$W(T^*)$ cost of MST

$W(M^*)$ cost of minimum weight matching, that is, the sum of edge length of all edges in the optimal matching

Because of the triangle inequality assumption,

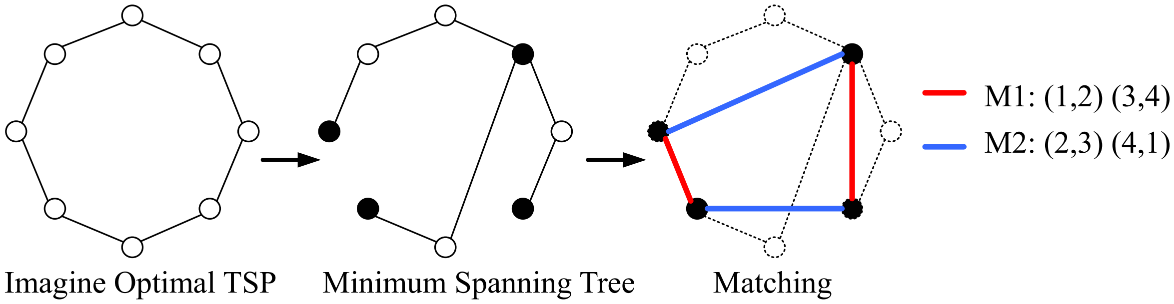
$$L^C \leq W(T^*) + W(M)$$

We already know that $W(T^*) \leq L^*$. It remains to show that $W(M^*) \leq \frac{1}{2} L^*$.

Index the nodes from odd degree matching in order of appearance in optimal TSP tour $(i_1, i_2, \ldots, i_m)$. Consider two feasible matching:

M1: (i_1, i_2) , (i_3, i_4) , … , (i_{m-1}, i_m)

M2: (i_2, i_3) , (i_4, i_5) , … , (i_m, i_1)



$$W(M^*) \le \frac{1}{2}[W(M_1) + W(M_2)]$$

And

$$W(M_1) + W(M_2) \le L^*$$

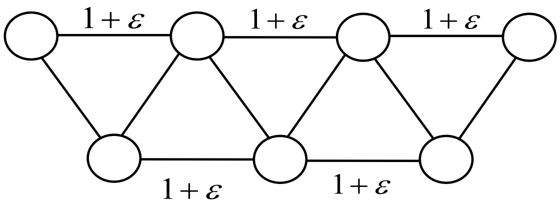
So,

$$W(M^*) \le \frac{1}{2} L^*$$

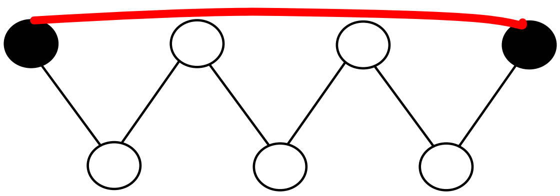
Thus,

$$L^* \le L^C \le W(T^*) + W(M^*) \le \frac{3}{2} L^* \qquad \blacksquare$$

Example



$$n = 7,\ L^* = 7$$



$$L^C = \frac{n-1}{2} + n \sim \frac{3}{2}$$

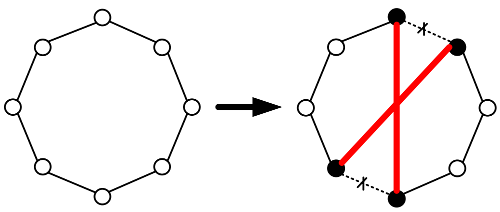
IMPROVEMENT HEURISTIC FOR TSP

Edge exchange Procedure

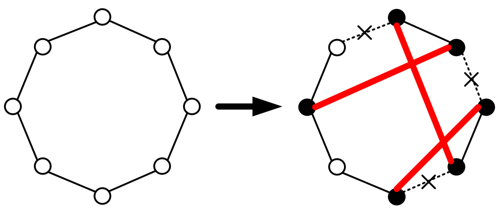
r-opt procedure

Exchange r-edges in solution with r-edges not in the solution

Example



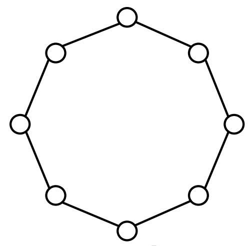
2-opt



3-opt

Lin and Kernighan (Variable r-opt)

Step1. Choose initial tour.

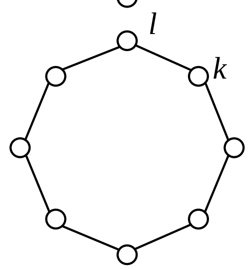


Step2. $G_0=0$

Select starting city l

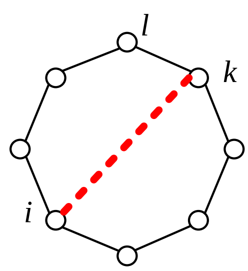
Consider edge (l , k) for removal

p=1

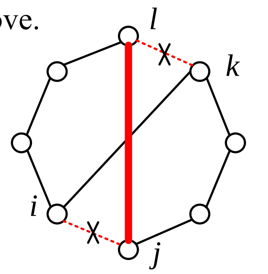


Step3. Choose edge (k , i) not in tour that maximize

$$g_1=C_{lk} - C_{ki} \quad \text{(Cost saving)}$$



Step4. (l , k) leaves, (k , i) enters, so edge adjacent to i must be remove.

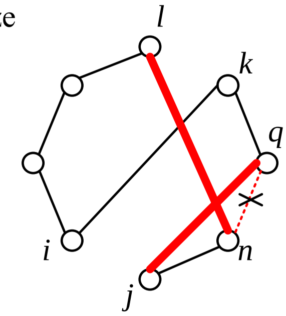


If we add (j , l), tour would be complete and total savings would be

$$G_1^* = g_1 + C_{ij} - C_{jl}$$

p=p+1

Step5. We don't necessarily add (j , l), instead, find q that maximize



$$g_p = C_{ij} - C_{jq}$$

so $$G_p = \sum_{s=1}^{p} g_s$$

$$G_p^* = G_p + C_{qn} - C_{nl}$$ where n is adjacent to q

This is the saving if we stopping here

$$G^* = \{ G_0^*, G_1^*, \dots \}$$

Step6. p=p+1

Stop if (1) no more subtour remain

(2) tour is found

(3) $G_p < 0$

(4) $G_p \le G^*$

else repeat Step5.

Final tour is one with best G^* from $G^* = \{ G_0^*, G_1^*, \dots , G_p^* \}$

Repeat using different starting cities

Chandra et al

$$\frac{L^{r-opt}}{L^*} \ge \frac{1}{4} n^{\frac{1}{2r}}$$

for an infinitely large family of problems that satisfy the triangle equality.

PROBABILISTIC ANALYSIS

Characterize performance if distribution assigned to input parameters.

Notes

(1) Provide insight

(2) typically asymptotic

(3) Often, we try to characterize expected value of optional solution

Example

Bin Packing Problem, we would like to show something like

$$\lim_{n\to\infty} \frac{E[b_n^*]}{n} = \frac{1}{2}$$

or something like

$$\lim_{n\to\infty} \frac{E[b_n^{NFD}]}{n} = 1.289$$