The Impact of Substitution and Intertemporal Demand on Coordinated Pricing-Inventory Strategies

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To coordinate pricing and inventory decision-making across time in a multi-product setting, firms must explicitly consider the impact of the price of a substitutable product in a particular period on both demand for a different product in that period (the substitution effect) and demand for that product in other periods (the intertemporal effect). We develop two demand models that explicitly consider these effects within a deterministic dynamic pricing and inventory ordering framework, and explore the impact of these effects on a firm's optimal pricing and inventory ordering decisions. We analyze pricing policies under various conditions, and in a computational study we explore both the impact of differing degrees of substitution and intertemporal effects on system performance in various settings, and the importance of explicitly considering these effects when making pricing and ordering decisions.

Key words: multi-product pricing; inventory; intertemporal demand; substitution

1. Introduction

A recent stream of operations management research has advocated the coordination of supply and pricing decisions to maximize supply chain efficiency and profit. The majority of this research however, has focused on the impact of the price of a product in a particular period on demand for that product in that period, and has ignored the effect of the price of other products in that period (which we call the substitution effect), and the price of that product in other periods (the intertemporal effect). Intuition would suggest, however, that these are often significant considerations in the real world. Clearly, some consumers may stockpile a product in anticipation of a price increase, wait to purchase a product in anticipation of a sale, or substitute a less expensive product if their initial choice is deemed to be too expensive.

Indeed, the marketing research literature provides compelling empirical evidence of the impact of intertemporal and substitution effects both individually and jointly on demand. For example, Draganska and Jain (2006) provides evidence of inter-product demand interactions and shows that firms exploit these interactions in various forms to discriminate customers. Pesendorfer (2002) analyzes price and sales data in a dynamic multi-product environment, and shows that demand for a partic-
ular product depends on both current and past prices of that product and its substitutes. These findings suggest that, depending on pricing decisions, customers may delay their purchases, (i.e., wait for the next sale), or change their preferences, (i.e., switch to other products). At the same time, there is empirical evidence that complications arise due to the lack of coordination of supply and pricing decisions. Corsten and Gruen (2003), and Taylor and Fawcett (2001) show that out-of-stock ratios for promoted items are significantly higher than ratios for non-promoted items. They also analyze the root causes of out-of-stock incidences, and list lack of coordination between pricing and inventory-ordering decisions as the biggest source of the problem.

Ahn et al. (2007) explores the impact of intertemporal demand effects when a monopolist firms sells a single product over a finite horizon. They consider a deterministic capacitated pricing and production model in the presence of intertemporal demand effects, and focus on determining the sequence of optimal prices in various scenarios. In this paper, we extend the results of Ahn et al. (2007) to the multi-product case, and utilize an analysis of this extended model to explore the impact of substitution and intertemporal effects on the effectiveness of dynamic pricing and inventory ordering strategies employed by firms, and to assess how the pricing of multiple products should be coordinated over time.

To do this, we consider a discrete time multi-period deterministic model of a monopolist that sells multiple substitutable products over a finite horizon. In each period, the firm coordinates pricing and inventory-ordering decisions for each product. Demand for each product in each period is a function of the price of that product in that period, the price of that product in previous periods, and price of other products in that period and in previous periods. We consider two versions of this model – in the initial version of the model we use a demand function more amenable to analysis, enabling us to develop a variety of analytical results. We then introduce a more detailed demand model, and use extensive computational experiments to explore which of insights from the simple model extend to this more detailed model.

We show that under many conditions, the firm can take advantage of most of the potential benefit of differentiating customers using intertemporal and substitution effects by charging only two prices for each product, a high price and a low price. Analyzing several special cases, we identify two factors that determine the timing decision and the range of optimal high and low prices: (i) the amount of available capacity and (ii) the degree of substitution among the products. In general, the additional

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1 As an example, Pesendorfer (2002) shows that, holding other variables constant, a one percent increase in last week’s product A’s (Heinz) price increases demand for product A by 2.1 percent. Moreover, an increase of product B’s (Hunts) price last week by one percent increases demand for A by one percent.

2 Corsten and Gruen (2003), in general, find a 2 : 1 ratio of promoted vs non-promoted out-of-stock rates.
value a firm can create by exploiting intertemporal and substitution effects is limited by the amount of excess capacity it possesses. The difference between high and low prices increases as excess capacity increases, regardless of the level of substitution. The degree of substitution, however, significantly impacts the sequence of pricing decisions, as well as the actual high and low prices. Specifically, when there is a great deal of possible substitution between products, a firm is typically better off offering simultaneous sales (that is, low prices) for multiple products and offering larger price reductions for the less expensive products, whereas when there is little substitution, alternating sales and large price reductions for the more expensive products leads to higher profit. We also explore conditions under which intertemporal and substitution effects can safely be ignored, and we find, among other things, that in general it is important to consider both of these effects, particularly substitution effects, and particularly when there is excess capacity.

Below, we briefly review the related literature and discuss our contribution in three areas: pricing with intertemporal demand models, pricing models with fixed initial inventory, and joint pricing and inventory-ordering models.

(i) Pricing with intertemporal demand models: There is a large body of research in marketing and economics focusing on intertemporal demand models, where demand is a joint function of current and past prices. Researchers in this area find that a firm facing this kind of demand function can increase profit by charging different prices over time, a marketing practice called intertemporal price discrimination. Conlisk et al. (1984) and Sobel (1984) analyze this type of demand model in monopolistic and oligopolistic settings, respectively. They show that firms can segment their customers over time by utilizing a cyclic pricing strategy. Assuncao and Meyer (1993) investigate pricing decision of a monopolist firm facing a demand function that depends on the interval between consecutive price promotions, and analyze the frequency of optimal price changes. Slade (1998) and Slade (1999) propose a demand model in which consumer goodwill increases as the firm keeps charging low prices, and analyze the resulting optimal pricing strategies. Extensive reviews of this literature can be found in Varian (1989) for monopolist firms, and in Stole (2007) for oligopolistic markets. Models developed in this literature focus on pricing decisions ignoring operational decisions. Our main objective in this paper, however, is to explicitly model this interaction and evaluate the impact of operational decisions in a multi-product environment.

(ii) Pricing models with single inventory replenishment: Papers in this area, often referred to as revenue management, consider dynamic pricing problems where a firm has a limited time to sell finite inventory and does not replenish inventory after the starting of the selling season. Review papers by McGill and van Ryzin (1999), Bitran and Caldentey (2003), and Elmaghraby and Keskinocak (2003) provide comprehensive surveys of this literature.
Recently, the revenue management papers featuring demand models that incorporate (i) time-based and (ii) product-based substitution effects have appeared. In the first category, Aviv and Pazgal (2008) study optimal mark-down pricing policies with two price levels, and analyze its impact on seller’s profit. Elmaghraby et al. (2008) consider a more general mark-down pricing model where seller can charge a finite number of price levels (instead of only two), and characterize optimal pricing policies under various modeling settings. The objective of papers in the second category is to model the impact of demand interactions among products on revenue management decisions. Papers in this category can be further divided into two subcategories. In the first one, seller is assumed to make assortment decisions without changing the prices (see, for example, Cachon and Kok (2007) and Aydin and Hausman (2008)) and in the second one, similar to our modeling setting, seller dynamically changes the prices for a fixed assortment (see, for example, Aydin and Porteus (2008), G. Bitran and Vial (2003), and L. Dong and Tian (2009)).

The majority of these research papers features the “no-replenishment” assumption on the supply side. Even though this assumption is valid for perishable products such as airline seats, hotel rooms, and seasonal items, in other industries firms have multiple replenishment opportunities during the selling season. In addition, most of the models in this literature focus on either intertemporal demand substitution in the single-product framework or inter-product demand substitution in the multiple-product framework. In contrast, we consider the interaction of both intertemporal demand effects and substitution effects, and analyze the impact of this interaction on both pricing and inventory-ordering decisions.

(iii) Pricing models with multiple inventory replenishment: In this stream, researchers consider joint dynamic pricing and inventory control under variety of assumptions, including no setup cost (Federgruen and Heching (1999)), non-zero setup cost (Chen and Simchi-Levi (2004), Huh and Janakiraman (2008)), lost sales (Chen et al. (2006)), and multiplicative demand uncertainty (Song et al. (2009)). Comprehensive surveys can be found in Elmaghraby and Keskinocak (2003) and Chan et al. (2004). Most of these models employ a demand function in which demand is affected only by current pricing decisions. Exceptions include Ahn et al. (2007) and Ahn et al. (2008), which assume a monopolistic firm selling a single product and analyze the impact of intertemporal demand interactions on pricing and inventory-ordering decisions. In contrast, we focus on multi-product demand models that capture both substitution and intertemporal interactions.

The remainder of this paper is structured as follows: In §2, we introduce the modeling framework, and analyze the model for uncapacitated and capacitated settings in §3 and §4, respectively. In §5, we present and analyze a more detailed model, and an algorithm for that model. In §6, we present results of computational testing. Finally, in §7, we discuss future research and conclude.
2. The Modeling Framework

We consider a multi-period deterministic inventory model of a monopolistic firm that sells \( M \) products over a finite number of time periods, \( T \). Let \( T = \{1, \ldots, T\} \), and \( M = \{1, \ldots, M\} \) denote set of time and product indices. At the start of each period, \( t \in T \), the firm places an order, \( x_i^t \leq q \), to its supplier for each product \( i \in M \), replenishes its inventory level, \( I_i^t \), and decides on its price, \( p_i^t \). We assume that \( P_i^t \) is the maximum price that can be charged for product \( i \) at period \( t \), and \( q \) is the maximum order size. Given a pricing policy \( p \), \( D_i^t(p) \) denotes total demand for product \( i \) at period \( t \).

We assume that ordering and holding costs are stationary in time and linear in quantities, and let \( c^i \) and \( h^i \) be the unit product and inventory-holding costs of product \( i \) at period \( t \), respectively.

The firm’s objective is to find an optimal pricing and inventory-ordering plan \((p^*, x)\) that maximizes total profit \((1a)\) subject to inventory balance \((1b)\), and constraints on the capacity \((1c)\) and decision variable bounds \((1d)\):

\[
\begin{align*}
(P, x) & \quad \max_{p,x} \sum_{i=1}^{M} \sum_{t=1}^{T} \left[ p_i^t D_i^t(p) - c^i x_i^t - h^i I_i^t \right] \\
\text{s. t.} & \quad x_i^t + I_i^{t-1} = D_i^t(p) + I_i^t \quad \text{for all } t \in T \text{ and } i \in M \\
& \quad \sum_{i=1}^{M} x_i^t \leq q \quad \text{for all } t \in T \\
& \quad 0 \leq p_i^t \leq P_i^t, \quad x_i^t \geq 0, \quad I_i^t \geq 0 \quad \text{for all } t \in T \text{ and } i \in M
\end{align*}
\]

To model demand associated with intertemporal and multiproduct effects, we consider in each period a demand function with three components, one representing the demand for each product in that period due to the price of that product (the standard linear demand function), one representing intertemporal demand (the demand in this period because the price for a product is lower than the price of that product in the previous period), and one representing interproduct demand due to demand that was originally for other products, both in this period and in the previous period. We utilize the following function, which captures these characteristics in a reasonable but relatively tractable way.

\[
D_i^t(p) = a^i - s^i p_i^t + (1 - \beta) \alpha s^i \left[ p_{i-1}^t - p_i^t \right]^+ + \beta \sum_{j=1}^{M} s^j I_{i \in m_t} = \sum_{j=1}^{M} s^j \left[ p_j^t - p_i^t \right]^+ + \sum_{j=1}^{M} \alpha s^j \left[ \min_{j=1 \ldots M} p_{j-1}^t - p_i^t \right]^+
\]

where \( m_t \) represents the indices of the lowest priced products, i.e., \( m_t = \arg \min_{i=1 \ldots M} p_i^t \), and \( I_{i \in m_t} = 1 \) if \( i \in m_t \), 0 otherwise.
The three components of the demand function are labeled in the equation above. Current demand for a product represents the demand generated by the product’s current price using a simple linear demand function. Most of the operations/pricing literature focuses on this type of demand.

In this paper, we consider two additional terms. The first of these represents the increase in demand in this period if price is lower for the product in this period than in the previous period. This product-specific intertemporal effect is captured by multiplying the price difference between this period and the previous period by the slope of the demand curve in this period (multiplied by a factor $\alpha$). This is the same function considered in Ahn et al. (2007).

The second additional term captures substitution effects, and consists of two sub-components. The first of these captures current substitution that results if this product is less expensive than alternative products in this period. Specifically, if this product is less expensive than other substitutable products in this period, this product is substituted for these other products by consumers with a lower reservation price for those other products. We model this for each product by multiplying the price difference by the slope of the demand curve for each of the higher-priced products. The second sub-component of this term captures demand for the product in this period if its price is lower than that of substitute products in the previous period. We multiply this term by $\frac{\sum_{j=1}^{M} I_{i \in m_t}}{\sum_{j=1}^{M} I_{i \in m_t}}$ to adjust for the possibility of multiple low price products in a given period.

We weight the product specific and non-specific demand components by $1 - \beta$ and $\beta$, respectively. Specifically, the inverse of $\beta$, i.e., $1 - \beta$, measures the relative weight of product specific effects and substitution effects (and is thus represents product loyalty/substitutability). This demand function has intuitively desirable characteristics: demand is decreasing in price in this period, increasing in the price difference (that is, how much lower) for this product between this period and the previous period, and increasing in the price difference (that is, how much lower) between this product in this period, and other products in this and the previous periods. Note that in this model, intertemporal effects are limited to a single period (so that demand in period $t$ is impacted by pricing decisions in period $t-1$, but not earlier. We do this to facilitate analysis and to develop insights – in Section 5, we extend this demand model to allow multiple previous periods’ pricing to impact demand in the current period.

We use the model described above to develop insights into the optimal multi-product pricing and inventory ordering/production policy in the presence of intertemporal demand effects and substitution effects, and to help us understand how system and demand characteristics and parameters impact the the optimal promotion and inventory policies in terms of:

- the appropriate range and number of different prices to offer,
• when and how often prices for products should be changed, and how price changes should be coordinated between products, and
• the impact of inventory and capacity-related constraints on pricing plans.

To answer the first two questions, we analyze an uncapacitated version of model \((P_s)\) in §3. Then, in §4, we consider the capacitated model and analyze a special case in order to characterize the impact of inventory and capacity-related constraints on pricing plans.

3. Analytical Results for the Uncapacitated Model

In this section, we consider a version of model \((P_s)\) with no constraint on the maximum order in each period (that is, in constraint \((1c)\), \(q = \infty\)), so that by replacing unit cost, \(c_i^t\), with \(\min_{t' \in \{1, \ldots, t\}} (c_{i'}^t + h_{i'}^t)\), we can transform the problem into an equivalent problem in which there is no incentive for the retailer to hold inventory. Thus, in each period the order quantity is equal to demand, (i.e., \(x_i^t = D_i^t(p)\)), so there is no inventory held. For the rest of this section, we assume that we have transformed the problem in this way. In §4, we consider a more complex setting with capacity limits in each period, which may in general lead to order quantities different than demand, and thus end-of-period inventory.

This model is difficult to analyze due to the combination of product-specific and substitution effects. To develop insight into the the impact of each of these effects, in the following subsections we consider them separately, of the model where \(\beta = 0\) and where \(\beta = 1\).

3.1. The Uncapacitated Model with No Substitution Effects:

Consider Model \((P_s)\) with \(q = \infty\) and \(\beta = 0\), so that there are no capacity constraints and no substitution effects. When \(\beta = 0\), there is no interaction between products, so model \((P)\) can be decomposed into \(M\) independent problems, each of which is analyzed in Ahn et al. (2007). The key observation in Ahn et al. (2007) is that any pricing plan (that is, sequence of prices to be charged over the horizon) can be decomposed into subsequences of decreasing prices (called price cycles). Using this observation, the authors develop an optimal (polynomial) algorithm built around determining prices for a price cycle of a given length, and then finding the optimal combination and sequence of price cycles of various lengths. In addition, when cost and demand function parameters are stationary over time, Ahn et al. (2007) explicitly characterize the optimal solution to the model:

**Proposition 1.** When \(\beta = 0\) and \(q = \infty\), model \((P_s)\) can be decomposed into \(M\) independent pricing problem, one for each product. For each product, the optimal pricing strategy involves alternating between high and low prices for every product in the market, where the high price for each product is given by:

\[
p_{hi}^t = \bar{p}^i + (\bar{p}^i - c^i) \left[ \frac{\alpha(\alpha + 2)}{4\alpha + 4 - \alpha^2} \right]
\]
and the low price is:

\[ p_{i0}^i = \hat{p}^i + (\hat{p}^i - c^i) \left[ \frac{\alpha(\alpha - 2)}{4\alpha + 4 - \alpha^2} \right] \tag{3} \]

where \( \hat{p}^i = a_i^s + c_i^s \).

Observe that in the context of this model, when there is no interaction between products, the full benefit of dynamic pricing strategy can be obtained by using only two price levels for each product. Chen et al. (2010) reached a similar conclusion with a related but different stochastic model.

### 3.2. The Uncapacitated Model with No Product-Specific Effects

To continue building insight, we next explore Model \((P_s)\) with \( q = \infty \) and \( \beta = 1 \), so that there are no capacity constraints and, in contrast to the previous section, no product-specific intertemporal effects, so that unmet demand “flows” to the minimum priced products. Before starting our analysis of the \( \beta = 1 \) case, observe that the first term in objective function (1a) in model \((P_s)\) is neither convex nor concave in \( p \) since it depends both on the level and the relative order of product prices. However, it turns out that the form of the objective function can be fully determined once we know which product has the minimum price at each period, and how these minimum prices change over time. Let \( p_{i\min}^t \) be the lowest price at period \( t \), i.e., \( p_{i\min}^t = \min_{j \in M} p_{jt}^i \). Let \( m_t \) be the set of product indices with lowest price at period \( t \). This set defines the following constraints on product prices in period \( t \):

\[ p_{i}^t = p_{i\min}^t \text{ for all } i \in m_t \text{ and } p_{i}^t \geq p_{i\min}^t \text{ for all } i \notin m_t \tag{4} \]

Let \( \pi_t \) be the fixed ordering constraint that imposes a particular order on the minimum prices with respect to time period \( t \), i.e.,

\[ p_{\pi_t}^{\min} \leq \ldots \leq p_{\pi_1}^{\min} \leq \ldots \leq p_{\pi_T}^{\min} \tag{5} \]

where \( \pi_t = \hat{t} \) if the minimum price at period \( \hat{t} \) (i.e., \( p_{\pi_{\hat{t}}}^{\min} \)) is the \( \hat{t} \)th lowest price among all \( T \) minimum prices. For a given \( m_t \) and \( \pi_t \), we show that:

**Proposition 2.** When \( \beta = 1 \), model \((P_s)\), together with the addition of linear constraints implied by \( m_t \) and \( \pi_t \) (i.e., constraints provided in (4) and (5), respectively), is a concave optimization problem.

Proofs for all Propositions and Lemmas are provided in Appendix A.

Using this proposition, we can generalize the “price cycle” concept presented above to the case when there is interaction between the demand for different products. More specifically, we consider the sequence made up of the minimum price over all products in each period, and define a **minimum price cycle** as a decreasing subsequence of this “minimum price sequence”, which leads to a well defined unit of analysis for characterizing the optimal dynamic pricing strategy when \( \beta = 1 \).

To begin the analysis, we prove that in the context of our model, it is not optimal for the firm (recall that the firm is a monopolist) to offer two products at the same lowest price.
Lemma 1. In the optimal solution to Model \((P_s)\) with \(\beta = 1\), in each period, there is only one product with the minimum price.

Lemma 1 implies that the firm will always segment customers by pricing products at different levels, a result that is consistent with the price differentiation literature. For example, in Draganska and Jain (2006), the authors empirically show that firms increase their profits by segmenting their customers based on multi-product pricing strategies.

Next, we explore how prices are ordered at a given time period \(t\), and how they change over the time. We first make the following assumption:

Assumption 1.

\[
\text{If } c_i^t \leq c_j^t \text{ then } a_i^t \leq a_j^t \text{ and } s_i^t \geq s_j^t
\]

Assumption 1 implies that if product \(j\)'s unit cost is greater than product \(i\)'s unit cost at period \(t\), then there are more customers who are willing to pay \(p\) or more for product \(j\) than for product \(i\).

Using Assumption 1, we prove that at any time, prices are ordered in the same way as costs.

Lemma 2. Suppose for model \((P_s)\) with \(q = \infty\) and \(\beta = 1\), Assumption 1 holds. Then, if \(c_i^t < c_j^t\), then in the optimal solution, \(p_i^t < p_j^t\).

Lemma 2 characterizes how prices are ordered in a given period, which in turn implies that the product with the minimum marginal cost at period \(t\) should also have the minimum price. We discuss an approach to determine the minimum price below, but first we use the observations above to express prices for all products in terms of the minimum price product, where the superscript \(\text{min}\) refers to the minimum price product:

Lemma 3. Consider model \((P_s)\) with \(q = \infty\) and \(\beta = 1\) under Assumption 1. Let \(p_{i}^{\text{min}}\) be the minimum price in period \(t\). Then, the optimal price for product \(i\) is given as follows:

\[
p_i^t = a_i^t - s_i^t(p_{i}^{\text{min}} - c_{i}^{\text{min}}) + s_i^t c_i^t
\]

It’s possible to use this result to develop a polynomial algorithm to find the optimal pricing strategy, even when demand and cost parameters are not stationary. However, for this case with stationary parameters, we can characterize optimal prices in closed form (in a parallel result to Proposition 1 for the model with only product-specific effects, although employing a different proof approach):

Proposition 3. Consider model \((P_s)\) with \(q = \infty\) and \(\beta = 1\) under Assumption 1. Let \(\tilde{p}_i^t\) be the optimal price for product \(i\) if the monopolist ignores all interactions between products and time periods,
so that, $\hat{p}_i = \frac{\hat{s}_i}{2\kappa} + \frac{c_i}{2}$. Then, the optimal prices for a price cycle $n$ that starts at period $f_n$ and ends at $l_n$ are

$$p^*_i = \begin{cases} c^{\min} + 2\kappa_i \left[ (1 + \hat{s}^{\min})(\hat{p}^{\min} - c^{\min}) + \sum_{j=1}^{M} \hat{s}_j (\hat{p}_j - \hat{p}^{\min}) \right], & \text{for } i = \min \vspace{1em} \\
\hat{p}_i + \kappa_i \left[ (1 + \hat{s}^{\min})(\hat{p}^{\min} - c^{\min}) + \sum_{j=1}^{M} \hat{s}_j (\hat{p}_j - \hat{p}^{\min}) \right], & \text{for } i \neq \min, \end{cases}$$

(6)

where $\hat{s}_i = \frac{\hat{s}_i}{\sum_{j=1}^{s_i} \hat{s}_j}$ and $\{\kappa_i\}_{i=1}^{n}$ is a positive decreasing sequence fully characterized in Appendix A.6.

Proposition 3 leads to several interesting observations: First, the optimal price for each product consists of two components: (i) a base price and (ii) a time-dependent adjustment. The base price for the lowest priced product (i.e., $i = \min$) is equal to its marginal cost, whereas for all the other products (i.e., $i \neq \min$), it is equal to the optimal non-interaction price (the optimal price if the monopolist ignores all interactions between products and time periods). This difference between base prices among the products creates a constant substitution effect from all the other products to the lowest priced product. Secondly, base prices are adjusted by time-dependent terms in order to account for intertemporal demand effects. Since these time adjustment terms are positive and decreasing in time, they lead to a decreasing pricing policy for each product. Finally, the depth of markdown between two consecutive periods for the lowest priced product is exactly twice as big as that of markdown for all the other products. This is primarily due to the fact that the intertemporal demand effects from other products are captured only by the lowest priced product and the magnitude of this demand component is proportional to the depth of the price markdown for the lowest-priced product. Therefore, in order to maximize the revenues from intertemporal demands, the optimal price for the lowest priced product fluctuates more aggressively than the prices of the other products.

Note too that since problem parameters are stationary, the total profit generated within a price cycle depends only on the cycle’s length, i.e., $\Pi_{f_n, l_n} = \Pi_{f_m, l_m}$ as long as $l_n - f_n = l_m - f_m$. Let $\Pi_L$ be the optimal total profit generated by a price cycle of length $L$, i.e., $\Pi_L = \Pi_{f_L}$, where $L = l - f + 1$. Then, it is trivial to show that $\Pi_L$ increases in $L$. In order to characterize the optimal price cycle length, we define $\bar{\Pi}_L = \Pi_L / L$, the average optimal profit per period generated by a price cycle of length $L$. In the next Proposition, we show that $L = 2$ maximizes $\bar{\Pi}_L$ for all demand and cost parameter values; in other words, a cyclic pricing policy with a cycle length of 2 periods maximizes the average profit:

**Proposition 4.** Consider model $(P_3)$ with $q = \infty$ and $\beta = 1$ under Assumption 1. The average cycle profit, $\bar{\Pi}_L$, is maximized by cycle length 2 for all demand and cost parameters.

Proposition 4 implies that for the infinite-horizon extension of the problem, the optimal pricing strategy involves alternating between high and low prices for every product in the market, where the high price for each product is given by:

$$p_{hi} = \begin{cases} c^{\min} + 2\left(\frac{2(\hat{s}^{\min} + 2 + 6\kappa)}{(\hat{s}^{\min} + 3 + 4\kappa)} - 4\kappa \right) (\hat{p}^{\min} - c^{\min})(\hat{s}^{\min} + 1) + \sum_{j=2}^{M} (\hat{p}_j - \hat{p}^{\min}) \hat{s}_j, & \text{for } i = \min \vspace{1em} \\
\hat{p}_i + \left(\frac{2\left(\hat{s}^{\min} + 2 + 6\kappa\right)}{(\hat{s}^{\min} + 3 + 4\kappa)} - 4\kappa \right) (\hat{p}^{\min} - c^{\min})(\hat{s}^{\min} + 1) + \sum_{j=2}^{M} (\hat{p}_j - \hat{p}^{\min}) \hat{s}_j, & \text{for } i \neq \min. \end{cases}$$

(7)
and the low price is:

\[ p_{i_{\text{lo}}} = \begin{cases} c_{\text{min}} + \frac{2(q_{\text{min}} + 3 + 2a_i)}{(q_{\text{min}} + 3)(q_{\text{min}} + 3 + 4a_i) - 4q_i} & \text{for } i = \min \hat{p} \\
\hat{p}^i + \frac{q_{\text{min}} + 3 + 2a_i}{(q_{\text{min}} + 3)(q_{\text{min}} + 3 + 4a_i) - 4q_i} \left[ \left( \hat{p}_{\text{min}} - c_{\text{min}} \right) (q_{\text{min}} + 1) + \sum_{j=2}^{M} (\hat{p}^j - \hat{p}_{\text{min}}) \bar{s}_j \right] & \text{for } i \neq \min. \end{cases} \] (8)

In a finite horizon problem instance, repeating this one period high- one period low pricing strategy is optimal when \( T \) is even. On the other hand, such a strategy would leave the last period uncovered when \( T \) is odd; therefore, we fine-tune the optimal pricing strategy for the final three periods.

**Proposition 5.** Suppose that the same conditions as in Proposition 4 hold. Then, for all demand and cost parameter values,

1. If the planning horizon length, \( T \), is an even number, then optimal product prices consist of 2-period price cycles and optimal total profit is equal to \( \frac{T}{2} \times \Pi_2 \).
2. On the other hand, if \( T \) is odd, then optimal product prices consist of 2-period price cycles for the first \( T - 3 \) periods, and a 3-period price cycle\(^3\) for the last three periods, and optimal total profit is equal to \( \frac{T-1}{2} \times \Pi_2 + \Pi_3 \).

Propositions 1 and 5 suggest that the effectiveness of a one period high - one period low pricing strategy is quite robust to the degree of interaction among products. In the next section, we restrict our attention to this specific pricing strategy, and explore how optimal high and low price levels change with respect to the degree of interaction between products, \( \beta \).

### 3.3. Sensitivity Analysis with respect to \( \beta \)

Propositions 1 and 5 characterize the the optimal dynamic pricing strategy for “extreme” special cases of model \((P_s)\) – only substitution effects, and only product specific intertemporal effects. In these cases, the optimal price cycle starts with a high price to skim high valuation customers, and then lowers the price in the second period. In this section, we explore the difference between high and low prices in the optimal strategy for each of these cases. Using the closed-form characterizations presented in Propositions 1 and 5 for \( \beta = 0 \) and \( \beta = 1 \), respectively, we compare the difference between high and low prices:

**Proposition 6.** Consider the optimal one period high - one period low pricing strategy for model \((P_s)\) with \( q = \infty \). Let \( \Delta_i^A \) and \( \Delta_i^R \) denote the absolute and relative difference between optimal high and low prices for product \( i \), i.e., \( \Delta_i^A = p_{hi} - p_{lo} \) and \( \Delta_i^R = \frac{p_{hi} - p_{lo}}{p_{hi}} \). Suppose that \( c^1 \leq c^2 \leq \cdots \leq c^M \) and \( \frac{a^1}{c^1} \leq \frac{a^2}{c^2} \leq \cdots \leq \frac{a^M}{c^M} \). Then,

- When \( \beta = 0 \), in the optimal two-price strategy, both absolute and relative difference between high and low prices are increasing in \( i \), i.e., \( \Delta_1^A \leq \Delta_2^A \leq \cdots \leq \Delta_M^A \) and \( \Delta_1^R \leq \Delta_2^R \leq \cdots \leq \Delta_M^R \).

\(^3\)The closed-form expression for a 3-period price cycle–\((p_{3,hi}, p_{3,me}, p_{3,lo})\)–is provided in Online Appendix A.6.
Figure 1  The change in absolute and relative differences between high and low prices for products 1 and 2 with respect to $\beta$. In this example, we assume that demand and cost parameters are stationary, i.e., $c^i_t = 15$, $s^i_t = 1$ for all $i \in \{1, 2\}$ and $t \in \{hi, lo\}$, and $a^1_t = 40, a^2_t = 30$ for all $t \in \{hi, lo\}$.

- When $\beta = 1$, in the optimal two price strategy, both absolute and relative difference between high and low prices are decreasing in $i$, i.e., $\Delta^A_1 \geq \Delta^A_2 \geq \ldots \geq \Delta^A_M$ and $\Delta^R_1 \geq \Delta^R_2 \geq \ldots \geq \Delta^R_M$.

Interestingly, this result demonstrates that substitution and product-specific effects have in some sense an opposite impact in pricing strategy. When there are no substitution effects ($\beta = 0$), the cheapest products have the smallest relative and absolute discounts between high and low price periods, whereas when there are no product-specific effects ($\beta = 1$), the cheapest products have the largest relative and absolute discounts between high and low price periods.

Although we are only able to analytically characterize optimal prices for these two extreme cases, our numerical study shows that as we might expect, when $0 < \beta < 1$, the direction of discounts falls between these two extremes. For example, in Figure 1, we consider a two product case, where optimal non-interaction prices are different (i.e., $\hat{p}^1 = 27.5$ and $\hat{p}^2 = 22.5$). In Figures 1(A) and 1(B), respectively, we present absolute (i.e., $\Delta^A_i = p^i_{hi} - p^i_{lo}$) and relative (i.e., $\Delta^R_i = \frac{p^i_{hi} - p^i_{lo}}{p^i_{hi}}$) difference between optimal high and low prices as $\beta$ changes between 0 and 1. Figure 1 illustrates that when $\beta$ is large (interaction between the two products is strong), the lower priced product has a larger price decrease than the higher-priced product. However, as the degree of interaction decreases ($\beta$ approaches 0), the higher-priced product’s price drop increases relative to that of the lower-priced product.

Intuitively, we observe this behavior because intertemporal demand affects different products differently, depending on the level of $\beta$. Also, as the degree of intertemporal demand effect in a product’s
demand function increases, the change in its price gets larger. Hence, in the $\beta = 1$ case, demand for the lowest-priced product is increased by intertemporal demand for all of the other products, which implies that the lowest-priced product has a more dramatic price change than the other products. As $\beta$ decreases, however, intertemporal demand shifts evenly between products, reducing the amount of price change for the lowest-priced product, and increasing the amount of price change for all other products.

4. Analytical Results for the Capacitated Model

When there is a capacity constraint in each period, it may be optimal to produce more than the demand. This implies that end-of-period inventory may no longer be zero, which in turn implies that our solution approaches based on decomposing the problem into independent sub-problems may not in general work. Furthermore, when capacity is limited, the timing of price changes can play a crucial role in lowering inventory holding costs.

For example, consider a two product case with identical demand and cost parameters for both products and no demand interaction between the products, so that $\beta = 0$. Now, any one period high-one period low pricing strategy can be implemented in two different ways: (i) the high and the low prices for both products can be offered at the same time (so that one period both products have high prices, and the next period both products have low prices – we call this the simultaneous strategy), or (ii) the high and the low prices can be offered at different times (so that in any period, one product has a high price and the other has the low price – we call this the alternating strategy). Even though both strategies lead to exactly the same demand realization for the firm, when capacity constraints are binding they are very different in terms of end-of-period inventory and hence inventory holding cost. The first strategy requires that the firm order more than demand when the price is high, and carry inventory to satisfy the higher demand in the lower price period. In contrast, the second strategy enables the firm to shift the allocation of capacity in each period between the two products more effectively, and does not require carried-over inventory between periods.

Thus, we can not decompose pricing decisions into “price cycles” when limited capacity may require inventory carryover between periods. Furthermore, in addition to the price levels, the specific timing of price changes needs to be explicitly determined in conjunction with inventory decisions. In order to better understand the impact of capacity constraints on the timing of price changes for different products, we restrict our attention to infinite horizon stationary one period high-one period low pricing strategies for two products. In what follows, we assume that both products have identical cost and demand parameters, i.e., $a^i = a$, $s^i = s$, $c^i = c$, and $h^i = h$ for $i = \{1, 2\}$ and consider the $\beta = 0$ and $\beta = 1$ cases separately. After these simplifications, for any pricing plan that consists of high and
low prices, it is trivial to show that total demand at each period follows a cyclic pattern. Because of this, inventory, if it is carried at all, is carried only from the low demand period (high price period) to the high demand period (low price period). Therefore, for this specific case, we can decompose the problem into an infinite series of the following two-period pricing and inventory ordering problems where inventory, if it is carried, is carried from periods \( t \) to \( t+1 \), and optimal prices for periods \( t \) and \( t+1 \), i.e., \( \mathbf{p} = (p_t^1, p_t^2, p_{t+1}^1, p_{t+1}^2) \) can be computed by solving the following math program:

\[
\begin{align*}
\max & \quad \sum_{i \in \{1,2\}} D_i^t(p_i)(p_i - c) + \sum_{i \in \{1,2\}} D_i^{t+1}(p_i^{t+1} - c) - hI \\
\text{s.t.} & \quad \sum_{i \in \{1,2\}} D_i^t(p) + I \leq q \\
& \quad \sum_{i \in \{1,2\}} D_i^{t+1}(p) - I \leq q \\
& \quad \mathbf{p} = (p_t^1, p_t^2, p_{t+1}^1, p_{t+1}^2) \in \{\mathbf{p}_s, \mathbf{p}_a\} \\
& \quad \mathbf{p}_l = (p_{hi}, p_{lo}, p_{hi}, p_{lo}), \quad l = s \\
& \quad (p_{hi}, p_{lo}, p_{hi}, p_{hi}), \quad l = a \\
& \quad I \geq 0; \quad 0 \leq p_{lo} \leq \frac{a}{s}, \quad \forall i \in \{1,2\}
\end{align*}
\]

For each \( l \in \{s, a\} \), the objective function can be shown to be a concave in \( \mathbf{p} \) and \( I \) (see the proof of Proposition 2 for details). The first two terms in the objective account for total profit generated by sales in periods \( t \) and \( t+1 \) and the last term accounts for the cost of inventory carried between \( t \) and \( t+1 \). The first and second constraints capture supply capacities and inventory carryover between \( t \) and \( t+1 \). Specifically, the first constraint ensures that sum of total demand realized in period \( t \) and the end-of-period inventory carried from \( t \) must be less than total capacity, whereas the second constraint ensures that total demand realized in period \( t+1 \) is less than total capacity plus carryover inventory from period \( t \). The third and fourth constraints specify either a simultaneous or an alternating strategy. The final two constraints specify non-negativity and ordering of the decision variables. Using Karush-Kuhn-Tucker (KKT) analysis, we can derive necessary and sufficient conditions for optimal high and low prices. The detailed analysis of these conditions is given Appendix A.9. The optimal high and low prices and inventory decisions that satisfy KKT conditions are provided in Table 1. Below, we discuss the optimal timing of high and low prices:

**Proposition 7.** Consider the infinite horizon stationary one period high- one period low pricing strategy for model (9) with stationary and identical cost and demand parameters.

- When \( \beta = 0 \), high and low prices for products 1 and 2 are offered in alternating fashion, i.e., \( \mathbf{p}^* = \mathbf{p}_a = (p_{hi}, p_{lo}, p_{hi}, p_{hi}) \), where \( p_{hi} \) and \( p_{lo} \) are provided in Table 1. Moreover, the optimal inventory associated with alternating pricing strategy is zero, i.e., \( I^* = 0 \).
Table 1  Optimal high and low pricing and inventory ordering decisions in a capacitated setting.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$p_{hi}^*$</th>
<th>$p_{lo}^*$</th>
<th>$I^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{q}{2} - \frac{1}{H_\alpha} \min(q, \bar{q})(2-\alpha)$</td>
<td>$\frac{q}{2} - \frac{1}{H_\alpha} \min(q, \bar{q}) \frac{2+\alpha - \alpha^2}{1+\alpha}$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{q}{2} - \frac{1}{H_\alpha} \min(q, \bar{q})(\alpha - 1)$</td>
<td>$\frac{q}{2} - \frac{1}{H_\alpha} \min(q, \bar{q}) \frac{2+\alpha - \alpha^2}{1+\alpha}$</td>
<td>$\min(q, \bar{q}) - q_1^* + \frac{3\alpha(1+\alpha)}{7+8\alpha - \alpha^2} - \min(q, \bar{q}) - q_2^*$</td>
</tr>
</tbody>
</table>

† Note that $\frac{q}{2}$ is the cut-off price, where demand becomes zero. The closed-form expressions for $\bar{q}$, $q_1$, $q_2$, $q_3$, and $q_4$ are provided in the Appendix A.9 and plotted in Figure 2.

Figure 2  The structure of optimal pricing and inventory-ordering policy.

(A) Regions for $\beta = 0$

(B) Regions for $\beta = 1$

- If $\beta = 1$, the high and low prices for products 1 and 2 are offered simultaneously. i.e., $p^* = p_s = (p_{hi}^*, p_{lo}^*, p_{lo}^*, p_{lo}^*)$, where $p_{hi}^*$ and $p_{lo}^*$ are provided in Table 1. Moreover, the optimal inventory associated with simultaneous pricing strategy is

$$I^* = [\min(q, q_2) - q_1]^+ + \frac{3\alpha(1+\alpha)}{7+8\alpha - \alpha^2} - [\min(q, q_3) - q_2]^+$$

Note that when $\beta = 0$, demand fluctuation is already attenuated due to the alternating nature of high and low prices, so inventory is always zero for each product. On the other hand, when $\beta = 1$, ...
inventory is carried over between high and low periods in order to smooth the demand fluctuations caused by the synchronized one period high-one period low pricing strategy.

Next, for each extreme value of $\beta$, we consider the sensitivity of pricing and inventory decisions with respect to capacity and holding cost. See Table 2 for the impact of these parameters on optimal decisions. Note that in the $\beta = 0$ case, optimal prices are decreasing in capacity, and constant in holding cost, whereas in the $\beta = 1$ case, both optimal prices and resulting inventory are non-monotone with respect to both holding cost and capacity. To succinctly express the relationship between prices and $h$ and $q$, we need to define one breakpoint for the $\beta = 0$ case (i.e., $\bar{q}$) and four breakpoints for the $\beta = 1$ case (i.e., $0 \leq q_1 \leq q_2 \leq q_3 \leq q_4$). These breakpoints and the structure of optimal policy with respect to capacity are plotted in Figure 2-a and 3-a for $\beta = 0$, and Figure 2-b and Figure 3-b, for $\beta = 1$, respectively. Note that these breakpoints define two and four regions in $h - q$ space for the $\beta = 0$ and $\beta = 1$ cases, respectively. We discuss these regions below.

Table 2 The impact of capacity and holding costs on optimal decisions.

<table>
<thead>
<tr>
<th>$\beta = 0$</th>
<th>$\beta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region 1</td>
<td>Region 1</td>
</tr>
<tr>
<td>Condition</td>
<td>Condition</td>
</tr>
<tr>
<td>$0 \leq q \leq \bar{q}$</td>
<td>$0 \leq q \leq q_1$</td>
</tr>
<tr>
<td>$\bar{q} &lt; q$</td>
<td>$q_1 &lt; q \leq q_2$</td>
</tr>
</tbody>
</table>

$\dfrac{\partial p_1^H}{\partial q}$ $\dfrac{\partial p_2^H}{\partial q}$ $\dfrac{\partial p_1^{LH}}{\partial h}$ $\dfrac{\partial p_2^{LH}}{\partial h}$  $\dfrac{\partial t^*}{\partial q}$ $\dfrac{\partial t^*}{\partial h}$

1 $- - 0 0 0 0$
2 $0 0 0 0 0 0$
3 $+ - 0 0 0 0$
4 $0 0 0 0 0 0$

* The closed-form expression for $\bar{h}$ is provided in the Appendix A.9 and is plotted in Figure 2. + and – correspond to positive and negative signs, respectively.

$\beta = 0$ case: Recall when $\beta = 0$, demand fluctuations generated by the price cycle for product 1 are balanced by product 2’s price cycle, which in turn eliminates the need for inventory carry-over and eliminates the impact of holding cost on pricing decisions. Therefore, the level of high and low prices is only governed by the available capacity $q$ in each period. As shown in Region 1 in Table 2 and in Figure 3-a, as capacity gets higher, both high and low prices decrease, but the difference between
them increases, which enables the firm to increase the intertemporal demand. Both prices reach to their unconstrained levels when \( q = \bar{q} \), and stay constant in \( q \) when \( q > \bar{q} \) in Region 2.

\( \beta = 1 \) case: Recall that a one period high - one period low price cycle allows a firm to skim high valuation customers by charging the high price first, and then collect additional low valuation customers in the second period. This creates a fluctuation in demand if high and low prices for both products are synchronized, and this fluctuation has an operational cost. Specifically, to meet demand fluctuations when capacity is limited, the firm may need to carry inventory from the high price to the low price period. Therefore, as the difference between high and low prices increases, the benefits from the skimming strategy increase, but so does the inventory level. This trade-off between the marketing benefit and the operational cost of these pricing strategies determines the behavior of optimal price and inventory levels in each region as follows (for the \( \beta = 1 \) case):

- Region 1: (High \( h \) and low \( q \)) In this region (see \( 0 \leq q < q_1 \) in Figure 3-b), the operational cost of carrying inventory dominates, so the firm chooses to charge the same constant prices for all periods. This practice is commonly employed in some industries in the form of an “Everyday Low Price” strategy. Note that the firm still differentiates products by charging different prices, but keeps prices constant over time.

- Region 2: (Low \( h \) and low \( q \)) In this region (see \( q_1 \leq q < q_3 \) in Figure 3-b), the benefit of the
price skimming strategy dominates the operational cost. Therefore, the firm holds positive inventory to shift some portion of the unused capacity to satisfy high demand during the low price period.

– Region 3: (Medium \( q \)) In this region (see \( q_3 \leq q < q_4 \) in Figure 3-b), even though the firm chooses to charge different high and low prices, it is not optimal to carry inventory between high and low periods. Note also that the firm shrinks its demand in the first period for the sake of high demand in the second period, even though doing so creates some unused capacity in the first period. This stems, unsurprisingly, from the fact that the benefit of differentiating customers over time dominates the opportunity cost of unused capacity.

– Region 4: (High \( q \)) In this region (see \( q_4 \leq q \) in Figure 3-b), capacity is not binding, so there is no need to carry inventory from low to high demand periods. Hence, in this region, optimal price is constant in both capacity and holding cost.

5. The Extended Model
Model \( (P_s) \) contains several simplifications. In particular, it assumes that cost, capacity and demand parameters are stationary, and that intertemporal effects are limited to a single period (so that demand in the current period is impacted by pricing decisions in the previous period, but not by pricing decisions in periods earlier than that). Below, we extend this model by extending demand function \( \bar{D}_i(t) \) as follows.

\[
\bar{D}_i(t) = a_i - s_i \frac{p_i^t}{q_t} + (1 - \beta) \sum_{u=1}^{K} \alpha_{s_{i-u}} \left[ \min_{i=1,...,u} p_i^{t-u} - p_i^t \right]^{+} + \beta \frac{I_{i \in m_t}}{I_{i \in m_t}} \sum_{j=1}^{M} s_i^j \left[ p_i^t - p_i^j \right]^{+} + \sum_{j=1}^{M} \sum_{u=1}^{K} \alpha_{s_{i-u}} \left[ \min_{i=1,...,M} p_i^{t-u} - p_i^t \right]^{+}
\]

where \( I_{i \in m_t} = 1 \) if \( i \in m_t \), 0 otherwise, and \( m_t = \arg \min_{i=1,...,M} p_i^t \). Observe that in this extended demand model, product-specific intertemporal effect is captured by multiplying the price difference between this period and the lowest price in some number \( u \) of previous periods by the slope of the demand curve in this period (multiplied by a factor \( \alpha \)), where \( 1 \leq u \leq K \). This is the same function considered in Ahn et al. (2007). Also, the second second sub-component of the substitution term now captures demand from up to \( K \) previous periods as appropriate. Finally, note that we weight the product specific and non-specific demand components by \( 1 - \beta \) and \( \beta \), respectively. Using this demand function, we can define the following optimization problem:

\[
(P_e) \quad \max_{p, x} \sum_{i=1}^{M} \sum_{t=1}^{T} \left[ p_i^t \bar{D}_i(t) - c_i x_i^t - h_i^t I_i^t \right] \quad \text{(10a)}
\]

s. t. \( x_i^t + I_i^{t-1} = \bar{D}_i(t) + I_i^t \quad \text{for all } t \in T \text{ and } i \in M \) \quad \text{(10b)}

\( \sum_{i=1}^{M} x_i^t \leq q_t \quad \text{for all } t \in T \) \quad \text{(10c)}

\( 0 \leq p_i^t \leq \frac{a_i^t}{s_i^t}, \quad x_i^t \geq 0, \quad I_i^t \geq 0 \quad \text{for all } t \in T \text{ and } i \in M \) \quad \text{(10d)}
As in Model (P_s), Model (P_e) finds optimal pricing and inventory-ordering plans \( p^* \) and \( x^* \) that maximize total profit (10a) subject to inventory balance equation (10b), capacity limitation (10c) and constraints on decision variables (10d).

5.1. An Optimal (Exponential) Algorithm For the Extended Model

Note that even special cases of Model (P_e) cannot be solved to optimality in the same way as special cases of Model (P_s). This is because capacity constraints make holding inventory necessary in some cases, and because the problem cannot be decomposed into price cycles in the same way, because price in a particular period can impact demand for more than one period in the future.

Nevertheless, we are able to develop an optimal exponential algorithm for the problem, which is sufficiently fast for our computational study. First, we note that the objective function of Model (P_e) is neither convex nor concave in \( p \) since it depends not only on the level, but also on the relative order of product prices. Therefore, first, we generalize Proposition 2 by showing that the first term in the objective can be transformed to a concave function in \( p \), provided that we add constraints that sort prices from the highest to the lowest. We call such constraints price sorting constraints. To define these constraints, we introduce a sorting function \( z \):

\[ z : n \in \{1, 2 \ldots T \times M\} \rightarrow (t_n, i_n) \in \{1 \ldots T\} \times \{1 \ldots M\}. \]

Using \( z \), we can define a sorting order on the prices such that that product \( i_n \)'s price at period \( t_n \) is the \( n^{th} \) lowest among \( T \times M \) prices in the pricing plan, i.e.,

\[ p_{i_1}^{t_1} \leq \ldots \leq p_{i_n}^{t_n} \leq \ldots \leq p_{i_{T \times M}}^{T \times M} \quad (11) \]

Let \( \mathbb{P}^z \) be the set of prices that respect the sorting implied by \( z \), i.e., \( p \in \mathbb{P}^z \) if and only if \( p \) satisfies the set of constraints implied by \( z \). Finally, let \( Z \) be the set of all sorting orders. Note that the cardinality of the set \( Z \) is equal to \((T \times M)!\). Using this notation, we prove the following proposition:

**Proposition 8.** The mathematical program given by (P_e), together with the addition of a set of price sorting constraints implied by \( z \in Z \), i.e., constraints provided in (11), is a concave optimization problem.

Proposition 8 suggests a generic algorithm for the extended demand model. This algorithm enumerates all possible price sorting constraints, solves a concave optimization problem for each one, and finds the optimal solution.

**Optimal Algorithm A1**

*Step 1.* Initialize \( \Pi^* = 0 \).
Step 2. For each price sorting constraint, $z \in Z$

Solve Model ($P_e$) with the addition of $p \in P^z$ to obtain optimal objective function value $\Pi^*(z)$, optimal prices $p^*(z)$ and order quantities $x^*(z)$.

If $\Pi^*(z) > \Pi^*$, then update $\Pi^* \leftarrow \Pi^*(z)$, $p^* \leftarrow p^*(z)$ and $x^* \leftarrow x^*(z)$

Step 3. Optimal pricing and inventory-ordering strategies are $p^*$ and $x^*$, respectively.

6. Computational Study

Up to this point, our analysis of special cases of our models has yielded closed-form expressions for prices under various conditions, as well as a variety of insights:

- Even in the presence of intertemporal and substitution effects, a limited number of price levels are necessary to maximize the benefit of dynamic pricing strategies (two prices, in many of the cases we consider).

- When substitution effects dominate, the difference between high and low prices is decreasing in product cost, whereas when product-specific intertemporal effects dominate, the difference between high and low prices increases in product cost.

- When there are multiple products, product-specific intertemporal effects encourage alternating sales to account for capacity constraints, whereas substitution effects encourage simultaneous sales and the use of inventory to account for capacity constraints.

- When capacity is binding and substitution effects dominate, the difference between high and low prices tends to be quite small, particularly at higher holding costs. As capacity increases and/or holding costs decrease, the difference between high and low prices increases. When product-specific intertemporal effects dominate, holding costs have little impact.

In general, based on this analysis, both product-specific intertemporal effects and substitution effects can impact optimal pricing policy, and capacity constraints can alter the relationship and relative impact of these effects. However, our analytical results are for various restricted versions of Model ($P_s$). In this section, we conduct a computational study using Model ($P_e$) in order to determine if these results generalize. In addition, managers may be inclined to focus exclusively on either product specific intertemporal effects or on current substitution effects when making decisions. We explore when this works relatively well, and when this can lead to bad pricing and inventory management decisions.

We complete a series of experiments by varying the problem parameters and employing several approaches to make pricing and inventory ordering decisions. First, we optimally solve the problem to explore the impact of system parameters on pricing and ordering decisions. Next, in order to explore the impact of explicitly considering product-specific intertemporal effects and substitution
efforts when making inventory and pricing decisions, we compare the profitability of approaches that explicitly consider none, one, or both of these effects (when both of them actually exist).

6.1. Experimental Design
To keep the number of instances at a manageable size, we consider two products and a four-period planning horizon (i.e., $M = 2$, and $T = 4$). We created a set of experiments by varying problem parameters as follows:

- Capacity Levels: We assume that total capacity in each period is constant and has one of the following three values: (i) Uncapacitated (equivalent to the case where $q = 100$), (ii) Mildly capacitated ($q = 50$), and (iii) Tightly capacitated ($q = 30$).
- Interaction Levels: We consider 3 different values for $K$: (K=1, 2, 3)
- Degree of substitution: We consider 3 different values for $\beta$: (i) Low ($\beta = \frac{1}{2}$), (ii) Med ($\beta = \frac{3}{4}$), (iii) High ($\beta = 1$)
- Demand Scenarios: For our computational study, we assume a base demand function $d(p) = a - sp$, where $a = 30$ and $s = 1$, and generate product 1 and 2’s current demand functions as follows: $d_1^t(p) = a - sp$, and $d_2^t(p) = \delta a - sp$ for all $t = 1 \ldots T$. We consider three different values for $\delta$: (i) Low differentiation ($\delta = 1$), (ii) Medium differentiation ($\delta = 1.1$) and (iii) High differentiation ($\delta = 1.2$)
- Production Costs: We use the following three values for the base production cost: (i) Low ($c = 0$), (ii) Medium ($c = 3$), and (iii) High ($c = 6$). Specific production cost for each product is obtained by multiplying base values with $\delta$ as follows: for product 1 and 2, $c_1^t = c$ and $c_2^t = \delta c$, respectively.
- Inventory Holding Costs: Similarly, we use the following three values for the base inventory-holding cost: (i) Low ($h = 0$), (ii) Medium ($h = 1$), (iii) High ($h = 4$). Specific inventory-holding cost for each product is obtained by multiplying base values with $\delta$ as follows: for product 1 and 2, $h_1^t = h$ and $h_2^t = \delta h$, respectively.
- Note: We let $\alpha = 1$ for all cases since the impact of varying $\alpha$ was explored in Ahn et al. (2007).

To summarize, we generate 729 instances by varying these parameters one at a time.

6.2. The Impact of System Parameters on Optimal Policy
For each set of parameter values, we compute the optimal policy using OPTIMAL ALGORITHM A1 described in §3.1. Note that this algorithm enumerates all possible orderings of price sequences and solves a concave maximization problem for each ordering. Since an optimization problem with a quadratic and concave objective function and a set of linear constraints is polynomially solvable with complexity of $O(T_C)$ and there can be potentially $(T \times M)!$ possible orderings, the time complexity of computing the optimal policy is $O((T \times M)!T_C)$. Let $\Pi^*$ denote the profit generated by optimal policy.
We make the following observations:

- As $K$, the number of periods in the past that can impact demand in the current period, increases, the number of distinct prices offered in the optimal policy also increases. More specifically, more than two prices are needed in order to construct optimal price cycles when $K > 1$. This essentially enables the firm to segment the intertemporal demand in a increasingly more refined fashion. However, we observe experimentally that the firm does not lose much by limiting pricing to two distinct prices per product instead of the optimal number of price points when $K > 1$. As as shown in Table 3, using a single constant price for each product leads to an average loss of 22.21% from the optimal profit, whereas as shown in Table 4, using two prices leads to an average loss of only 3.21% from the optimal profit.

- The difference between the maximum and minimum prices in a price cycle follows the same pattern as we observed in the 2-period price cycle case. Specifically, the difference between the maximum and minimum prices for the product with the lowest cost increases in $\beta$, whereas that for the product with the highest cost decreases. Also, the minimum prices in a price cycle and the maximum prices in a price cycle occur at the same time for all the products when $\beta = 1$ (i.e., no product specific intertemporal effects exist). However, as $\beta$ decreases, this pattern of simultaneous high and low prices gets perturbed to reduce the demand volatility over time, i.e., minimum (and maximum) prices for products are offered at different times to smooth the total demand realization across time.

- Finally, as the available capacity gets tighter or inventory holding cost increases, the difference between all of the different prices, both over time and over products, decreases, so that all the prices converge to each other. In effect, the the firm is focusing on current demand, since either there is not enough extra capacity or it is too expensive to take advantage of substitution and intertemporal effects.

6.3. The Value of Considering Substitution and Intertemporal Effects

To explore the value of explicitly considering product-specific intertemporal effects and substitution effects when making inventory and pricing decisions, we compare the profitability of approaches that explicitly consider none, one, or both of these effects. To do this, we define three heuristics for Model $(P_e)$. This heuristics work by optimally solving simplified versions of Model $(P_e)$ to find pricing and ordering policies. In other words, these simplified versions ignore either product-specific intertemporal effects, or current substitution effects, or both, and determine what the optimal pricing and ordering policy would be. We then assess the performance of these heuristics in the presence of of the product-specific intertemporal effects and current substitution effects that they didn’t consider.
Table 3  Percentage of decrease from the optimal profit by restricting price cycles to be of length 1, i.e., using constant pricing policies

<table>
<thead>
<tr>
<th>Capacity</th>
<th>Low Diff</th>
<th>Med Diff</th>
<th>High Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K = 1$</td>
<td>$K = 2$</td>
<td>$K = 3$</td>
</tr>
<tr>
<td></td>
<td>$K = 1$</td>
<td>$K = 2$</td>
<td>$K = 3$</td>
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<tr>
<td></td>
<td>$K = 1$</td>
<td>$K = 2$</td>
<td>$K = 3$</td>
</tr>
<tr>
<td>Uncap</td>
<td>21.43%</td>
<td>29.09%</td>
<td>35.55%</td>
</tr>
<tr>
<td>Med</td>
<td>17.23%</td>
<td>20.86%</td>
<td>23.57%</td>
</tr>
<tr>
<td>Tight</td>
<td>8.59%</td>
<td>9.91%</td>
<td>11.07%</td>
</tr>
<tr>
<td>Average</td>
<td>15.93%</td>
<td>20.21%</td>
<td>23.71%</td>
</tr>
<tr>
<td>Prod cost</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>15.01%</td>
<td>18.89%</td>
<td>22.20%</td>
</tr>
<tr>
<td>Med</td>
<td>16.13%</td>
<td>20.45%</td>
<td>23.96%</td>
</tr>
<tr>
<td>High</td>
<td>17.09%</td>
<td>21.91%</td>
<td>25.69%</td>
</tr>
<tr>
<td>Average</td>
<td>15.93%</td>
<td>20.21%</td>
<td>23.71%</td>
</tr>
<tr>
<td>Inv cost</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>17.27%</td>
<td>22.82%</td>
<td>27.26%</td>
</tr>
<tr>
<td>Med</td>
<td>16.31%</td>
<td>20.42%</td>
<td>24.12%</td>
</tr>
<tr>
<td>High</td>
<td>14.22%</td>
<td>17.40%</td>
<td>19.76%</td>
</tr>
<tr>
<td>Average</td>
<td>15.93%</td>
<td>20.21%</td>
<td>23.71%</td>
</tr>
</tbody>
</table>

6.3.1. The Heuristic Policies: Specifically, for our heuristic policies, we first solve each of our instances of Model (P_e) with either $\alpha$ set equal to zero and $\beta$ set equal to one for all instances, or $\beta$ set equal to zero, and $\alpha$ set equal to one for all instances, or both set equal to zero for all instances. Solving these restricted versions of Model (P_e) gives us a set of prices. Given these prices, we then evaluate demand given the actual values of $\alpha$ and $\beta$ for each instance. Since the prices we determined are calculated using our restricted demand functions, we may not have enough capacity to meet this demand. Therefore, we solve an LP to determine the final inventory decisions to optimally allocate capacity given the prices we have determined and meeting demand up to this demand. We call our heuristic policies the $\alpha = 0$ heuristic, the $\beta = 0$ heuristic, and the $\alpha \beta = 0$ heuristic, respectively, and let $h = \{\alpha = 0, \beta = 0, \alpha \beta = 0\}$ be the indices for these policies. We detail each of these heuristic policies below:

**Heuristic Policy $h$**

1. Let $P_h = \{P_i\}_{i=1}^T$ be the heuristic pricing plan found by solving Model (P_e) under a restricted demand model $D_h$, where $D_h, h = \{\alpha = 0, \beta = 0, \alpha \beta = 0\}$ is obtained by setting, respectively, $\alpha = 0$ and $\beta = 1$, or $\alpha = 1$ and $\beta = 0$ or both $\alpha = 0$ and $\beta = 0$ in demand function.
Table 4  Percentage of decrease from the optimal profit by restricting price cycles to be of length 2, i.e., using high-low pricing policies

<table>
<thead>
<tr>
<th></th>
<th>Low Diff</th>
<th>Med Diff</th>
<th>High Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>K = 1</strong></td>
<td>0% 6.31%</td>
<td>11.63%</td>
<td>0% 5.84%</td>
</tr>
<tr>
<td><strong>K = 2</strong></td>
<td>0% 6.06%</td>
<td>11.19%</td>
<td>0% 10.79%</td>
</tr>
<tr>
<td><strong>K = 3</strong></td>
<td>0% 2.53%</td>
<td>4.52%</td>
<td>0% 3.81%</td>
</tr>
<tr>
<td><strong>Capacity</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uncap</td>
<td>0% 3.16%</td>
<td>5.41%</td>
<td>0% 2.09%</td>
</tr>
<tr>
<td>Med</td>
<td>0% 1.21%</td>
<td>2.28%</td>
<td>0% 0.87%</td>
</tr>
<tr>
<td>Tight</td>
<td>0% 1.03%</td>
<td>1.94%</td>
<td>0% 1.66%</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>0% 3.69%</td>
<td>6.71%</td>
<td>0% 3.12%</td>
</tr>
<tr>
<td><strong>Prod cost</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>0% 3.38%</td>
<td>6.26%</td>
<td>0% 2.94%</td>
</tr>
<tr>
<td>Med</td>
<td>0% 3.72%</td>
<td>6.75%</td>
<td>0% 3.07%</td>
</tr>
<tr>
<td>High</td>
<td>0% 4.11%</td>
<td>7.34%</td>
<td>0% 3.43%</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>0% 3.69%</td>
<td>6.71%</td>
<td>0% 3.12%</td>
</tr>
<tr>
<td><strong>Inv cost</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>0% 4.73%</td>
<td>8.52%</td>
<td>0% 3.97%</td>
</tr>
<tr>
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<td>0% 3.53%</td>
<td>6.71%</td>
<td>0% 3.04%</td>
</tr>
<tr>
<td>High</td>
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<td>4.85%</td>
<td>0% 2.32%</td>
</tr>
<tr>
<td><strong>Average</strong></td>
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<td>6.71%</td>
<td>0% 3.12%</td>
</tr>
<tr>
<td><strong>β</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td>0% 4.07%</td>
<td>7.50%</td>
<td>0% 3.49%</td>
</tr>
<tr>
<td>Med</td>
<td>0% 3.69%</td>
<td>6.72%</td>
<td>0% 3.10%</td>
</tr>
<tr>
<td>High</td>
<td>0% 3.33%</td>
<td>5.95%</td>
<td>0% 2.79%</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>0% 3.69%</td>
<td>6.71%</td>
<td>0% 3.12%</td>
</tr>
</tbody>
</table>

2. Calculate the true demand realization using the true demand function for each period and product generated by the heuristic pricing plan as follows: \( \hat{D}_{i,h;t} = D_i(t(p_h)) \).

3. Solve the following linear program to optimally allocate capacity:

\[
\Pi_m = \max \sum_{t=1}^{T} \sum_{i=1}^{M} p_i d_{i,h,t} - \sum_{t=1}^{T} \sum_{i=1}^{M} c_i x_i - \sum_{t=1}^{T} \sum_{i=1}^{M} h_i I_{i,h,t} \\
\text{subject to} \quad x_i + I_{i,h,t-1} = d_{i,h,t} + I_{i,h,t} \\
\quad x_i \leq Q_t \\
\quad d_{i,h,t} \leq D_{i,h,t} \\
\quad x_i, d_{i,h,t}, I_{i,h,t} \geq 0 \quad \forall t = 1, \ldots, T \text{ and } i = 1, \ldots, M
\]

where \( I_{i,h,t} \) and \( d_{i,h,t} \) denote the inventory level and amount of satisfied demand for each product \( i \) at each period \( t \) under a heuristic policy \( h \).

In the computational study, we compare these policies with respect to their operational performance. Recall that \( \Pi^\ast \) is the optimal profit and define average inventory under the optimal solution \( \bar{I}^\ast = \frac{\sum_{t=1}^{T} \sum_{i=1}^{M} I_{i,h,t}}{M \times T} \). To compare the performance of the optimal policy with that of the heuristics, we define percentage improvement in profit for the optimal policy over that of each heuristic \( h \) as follows:

\[
\Delta_h = \frac{\Pi^\ast - \Pi_h}{\Pi_h} \times 100
\]
Also, we denote average inventory under a heuristic policy by:

\[
\bar{I}_h = \frac{\sum_{t=1}^{T} \sum_{i=1}^{M} I_{h,t}}{M \times T}
\]

and average stock-out due to our heuristic demand model assumptions under heuristic \( h \) by:

\[
\bar{I}^-_h = \frac{\sum_{t=1}^{T} \sum_{i=1}^{M} \left[ \hat{D}_{h,t} - d_{h,t} \right]}{M \times T}
\]

The results for \( \Delta_h \), \( \bar{I}^* \), \( \bar{I}_h \) and \( \bar{I}^-_h \) are presented in Table 6, Table 7, and Table 8, respectively.

6.3.2. Observations: We make the following observations:

- On average, the gain from using the optimal policy with respect to all other heuristic policies is quite significant. The average gain is 15.28%.
  - In general, the gain from using the optimal policy decreases as the capacity becomes more constrained. This is intuitive since when the capacity is tight, the firm can only satisfy the current demand. Therefore, ignoring either temporal effect or substitution effect or both does significantly hurt the firm’s profits.
  - In general, the gain from using the optimal policy increases as \( K \) and the degree of differentiation between product demand functions (i.e., \( \delta \)) increase. This is because as either \( K \) or \( \delta \) increases, temporal and substitution effects become more significant, and can be exploited to increase profit more significantly. Conversely, failing to take either one of these effects into account has increasing negative impact.
  - In general, the gain from using the optimal policy increases as product and inventory-holding costs increase. This is because both overage and underage values increase as product and inventory costs increase, and hence the firm is penalized more by supply-demand mismatches.

- Comparing the performance of the heuristic policies to the optimal policy, we notice that correctly accounting for substitution effects is in general more important than correctly accounting for temporal effects. On average, focusing only on substitution effects (i.e., following an \( \alpha = 0 \) policy) leads to approximately a 10% reduction in profits, whereas focusing only on temporal effects (i.e., following a \( \beta = 0 \) policy) leads to a %14 reduction. Ignoring both factors leads to an average of a %21.2 decrease in profits. The underlying reason for the difference between \( \alpha = 0 \) and \( \beta = 0 \) policies can be seen by referring to the average overstock and understock performances of these policies, as provided in Table 7 and Table 8 respectively. Note that since the \( \beta = 0 \) policy overemphasizes temporal effects, it leads to excessive fluctuations in demand, and ultimately to supply-demand mismatches. Indeed, observe in Table 7 and Table 8 that the \( \beta = 0 \) policy has both the highest average inventory and the highest
number of stock-outs among all heuristic strategies. In contrast, the $\alpha = 0$ policy overemphasizes substitution effects, leading to a more stable demand pattern, which is from an operational viewpoint less costly to satisfy. In fact, in Tables 7 and 8 we see that $\bar{I}_{\alpha=0} = I_{\alpha=0} = 0$ for all of the cases.

These observations also reinforce our analysis of the $\beta$ parameter, where we showed that intertemporal pricing policy is substantially impacted by the degree of substitution effect. In general, it is apparent that although intertemporal pricing is widely employed, it is essential to account for substitution effects when making intertemporal pricing decisions. Otherwise, there is a risk of potential losses due to excessive supply-demand mismatches.

- Specifically, the $\beta = 0$ policy performs very well when products are quite similar ($\delta = 1$) and quickly degrades as products get increasingly differentiated. Again, as the degree of differentiation among products increases, product substitution effects become more important, so that ignoring these effects leads to operational inefficiencies in the form of supply-demand mismatches. As shown in Table 8, the number of stock-outs quickly builds up as $\delta$ increases.

- In contrast, the performance of the $\alpha = 0$ policy improves as $\delta$ increases. Similarly, as inventory holding cost increases, the profitability of this policy approaches that of the optimal policy. This is because the high holding cost environment decreases the difference between the high and low prices in the optimal pricing policy, which is consistent with the $\alpha = 0$ policy.

7. Conclusions and Future Research

In this paper, we consider a set of models that explicitly considers both intertemporal and product-based substitution effects in a multi-product inventory ordering and pricing framework. Through a combination of analysis and computation, we discovered that the firm can typically exploit most of the benefits of intertemporal and substitution effects using only two prices a high and low (or regular and sale) price. A detailed analysis of policies in which firms alternate between low and high prices suggests that the timing and range of high and low prices is significantly impacted by the excess capacity of the firm and the amount of substitution between products.

Our findings are summarized in Table 5. In general, the additional value a firm can create by exploiting intertemporal and substitution effects is limited by the amount of excess capacity it possesses, which is reflected by an increase in the optimal difference between high and low prices as excess capacity increases. The degree of substitution also significantly impacts the sequence of pricing decisions, as well as the actual high and low prices. Specifically, when there is a great deal of possible substitution between products, a firm is typically better off offering simultaneous sales (that is, low prices) for multiple products and offering larger price reductions for the less expensive products, whereas when there is little substitution, alternating sales and large price reductions for the more
### Table 5 Summary of results.

<table>
<thead>
<tr>
<th></th>
<th>Low Excess Capacity</th>
<th>High Excess Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Little Substitution</td>
<td>Constant pricing or alternating high and low prices</td>
<td>Alternating or simultaneous high and low prices larger price reduction for high cost product</td>
</tr>
<tr>
<td>High Substitution</td>
<td>Constant pricing or simultaneous high and low prices</td>
<td>Simultaneous high and low prices larger price reduction for low cost product</td>
</tr>
</tbody>
</table>

expensive products leads to higher profit. Through extensive numerical studies, we showed that these results can be extended to more general settings, and that in general it is important to consider both substitution and intertemporal effects, particularly substitution effects, and particularly when there is excess capacity.

Our models have a variety of limitations. We assume that firm’s demand is not affected by its competitors’ prices, and that demand functions are deterministic. Nevertheless, we think that the models and results presented in this paper are a good initial step toward capturing the impact of both substitution and intertemporal effects in a multi-product setting where ordering and pricing decisions need to be made.

Building on our initial framework, we are currently extending these models in several ways. We are considering models with fixed set-up costs associated with inventory orders. We can relatively easily integrate fixed set-up costs into the modeling framework developed in this paper, and this extension will help us to explore the joint impact of marketing and operational factors on pricing and ordering synchronization. We are also working to extend our models to a game-theoretic multi-firm setting where each firm sells one or more products and demand for each product is a function of its own price and the prices of other products. Finally, we hope to extend our models to more complex demand functions, including those that model stochastic demand.

### References


Table 6  Percentage of profit loss due to ignoring temporal, and substitution effects.

<table>
<thead>
<tr>
<th>Capacity</th>
<th>Low Diff ((\delta = 1))</th>
<th>Medium Diff ((\delta = 1.1))</th>
<th>High Diff ((\delta = 1.2))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\Delta_{\beta=0})</td>
<td>(\Delta_{\alpha=0})</td>
<td>(\Delta_{\alpha,\beta=0})</td>
</tr>
<tr>
<td>Uncap</td>
<td>5.20%</td>
<td>17.99%</td>
<td>28.69%</td>
</tr>
<tr>
<td>Med</td>
<td>6.34%</td>
<td>11.23%</td>
<td>20.99%</td>
</tr>
<tr>
<td>Tight</td>
<td>3.50%</td>
<td>4.89%</td>
<td>10.07%</td>
</tr>
<tr>
<td>Average</td>
<td>5.01%</td>
<td>11.37%</td>
<td>19.91%</td>
</tr>
<tr>
<td>K</td>
<td>1</td>
<td>5.54%</td>
<td>7.66%</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.88%</td>
<td>11.62%</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4.61%</td>
<td>14.83%</td>
</tr>
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<td>5.01%</td>
<td>11.37%</td>
<td>19.91%</td>
</tr>
<tr>
<td>Prod cost</td>
<td>Low</td>
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<tr>
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<td>11.37%</td>
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</tr>
<tr>
<td>Inv cost</td>
<td>Low</td>
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</tr>
<tr>
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</tr>
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<td>High</td>
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<td>8.71%</td>
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<tr>
<td>Average</td>
<td>5.01%</td>
<td>11.37%</td>
<td>19.91%</td>
</tr>
</tbody>
</table>


Table 7  Average inventory levels realized under each model.

<table>
<thead>
<tr>
<th></th>
<th>Low Diff ((\delta = 1))</th>
<th>Medium Diff ((\delta = 1.1))</th>
<th>High Diff ((\delta = 1.2))</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>(\bar{I}^*) (I_{\alpha=0}) (I_{\alpha\beta=0})</td>
<td>(\bar{I}^*) (I_{\alpha=0}) (I_{\alpha\beta=0})</td>
<td>(\bar{I}^*) (I_{\alpha=0}) (I_{\alpha\beta=0})</td>
</tr>
<tr>
<td>Capacity</td>
<td>Uncap</td>
<td>Med</td>
<td>Tight</td>
</tr>
<tr>
<td></td>
<td>0.00 0.00 0.00 0.00</td>
<td>10.12 10.62 0.00 0.00</td>
<td>6.96 7.42 0.00 0.00</td>
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<td>6.38 8.30 0.00 0.00</td>
<td>6.38 8.30 0.00 0.00</td>
</tr>
<tr>
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<td>5.65 7.91 0.00 0.00</td>
<td>5.56 9.30 0.00 0.00</td>
</tr>
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<td>5.65 7.91 0.00 0.00</td>
<td>5.56 9.30 0.00 0.00</td>
</tr>
<tr>
<td>Prod cost</td>
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<td>5.81 7.90 0.00 0.00</td>
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<tr>
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<td>5.65 7.91 0.00 0.00</td>
<td>5.56 9.30 0.00 0.00</td>
</tr>
<tr>
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<td>1.79 4.71 0.00 0.00</td>
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<td>5.65 7.91 0.00 0.00</td>
<td>5.56 9.30 0.00 0.00</td>
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<td>5.95 7.26 0.00 0.00</td>
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<tr>
<td>Average</td>
<td>5.69 6.01 0.00 0.00</td>
<td>5.65 7.91 0.00 0.00</td>
<td>5.56 9.30 0.00 0.00</td>
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Table 8  Average amount of stock-outs due to ignoring temporal and substitution effects in demand.

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<tr>
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<th>Medium Diff ($\delta = 1.1$)</th>
<th>High Diff ($\delta = 1.2$)</th>
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<td>11.66 0.00 9.90</td>
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Appendix A: Proofs of Propositions and Lemmas

A.1. Proofs of Proposition 2 and Proposition 8:

Since Proposition 2 is a special case of Proposition 8, here, we only provide a proof for Proposition 8.

Fixed ordering constraints \( z \in Z \) completely specify the lowest priced products at each period \( t, j(t) \), and their order with respect to time, \( \pi(t) \). Hence, we will prove that Model (P) is a concave maximization problem under linear constraints implied by \( j(t) \) and \( \pi(t) \).

In order to show that the objective function is concave for all feasible prices in Model (P), it is enough to consider only revenue terms, since the production and holding costs are linear and concave. Let \( p = \{p_t^i|i = 1,\ldots,M,\text{ and }t = 1,\ldots,T\} \) be a pricing plan. Using the fact that demand function is convex combination of \( D_t^{\beta=0,i}(p) \) and \( D_t^{\beta=1,i}(p) \), we can write the revenue function as follows:

\[
\text{(13)}
\]

where

\[
D_t^{\beta=0,i}(p) = a_t^i - s_t^i p_t^i + \sum_{k=1}^{\min(K,t-1)} \alpha^k s_{t-k} \left( \min_{u=t-k,...,t-1} p_u^{(i)} - p_t^i \right) + \sum_{k=1}^{M} s_t^i (p_t^i - p_t^{(i)}) \quad \text{if } i = j(t); \quad \text{o.w.}
\]

and

\[
D_t^{\beta=1,i}(p) = a_t^i - s_t^i p_t^i + \sum_{k=1}^{\min(K,t-1)} \alpha^k s_{t-k} \left( \min_{u=t-k,...,t-1} p_u^{(i)} - p_t^i \right) + \sum_{k=1}^{\min(K,t-1)} \alpha^k s_{t-k} \left( \min_{u=t-k,...,t-1} p_u^{(i)} - p_t^i \right)
\]

Ahn, Güümüş and Kaminsky (2007) show that \( \sum_{t=1}^{M} \sum_{i=1}^{T} p_t^i \left( D_t^{\beta=0,i}(p) \right) \) is concave in \( p \). Hence, it suffices to show that \( \sum_{t=1}^{M} \sum_{i=1}^{T} p_t^i \left( D_t^{\beta=1,i}(p) \right) \) is concave in \( p \).

For notational simplicity, we define \( p_{(t,k,i)} = \begin{cases} p_t^i & \text{if } k = 0; \\ \min_{u=t-k,...,t-1} p_u^{(i)} & \text{if } k > 0. \end{cases} \) and

\[
I_{(t,k,i)} = \begin{cases} 1, & \text{if } k = 0 \text{ and } i = j(t); \\ 1, & \text{if } k > 0 \text{ and } p_{(t,k,i)} \leq \min_{u=t-k,...,t-1} p_u^{(i)}; \\ 0, & \text{o.w.} \end{cases}
\]

Also let \((t,k,i)\) and \((l,k,i)\) be the time and product indices of \( p_{(t,k,i)} \). Finally, we let \( \alpha^0 = 1 \) and combine last two terms in \( D_t^{\beta=0,i}(p) \) by taking the summation on \( k \) from 0 to \( \min(K,t-1) \) as follows:

\[
D_t^{\beta=0,i}(p) = a_t^i - s_t^i p_t^i + \sum_{l=1}^{\min(K,t-1)} \sum_{k=0}^{M} \alpha^k s_{l-k} \left( p_{(t,k,i)} - p_t^i \right) I_{(l,k,i)}
\]

In order to show that \( f(p) \) is concave in \( p \in F \), it suffices to prove that for any fixed \( \hat{p} \in F \), the tangent line at \( f(\hat{p}) \) always lies above \( f(p) \) for all \( p \in F \). In the next Lemma, we prove this. Therefore, \( f(p) \) is concave over \( F \), so that for any particular set of fixed ordering constraints, the corresponding problem to determine an optimal price plan becomes a concave maximization problem with the set of linear constraints.

**Lemma 4.** Under any set of fixed ordering constraints \( F \), \( f(p) - f(\hat{p}) \leq \nabla f(\hat{p})(p - \hat{p}) \) for all \( p \in F \).

**Proof of Lemma 4.** From equation (13), \( f(p) - f(\hat{p}) \) can be expressed as follows:

\[
\begin{aligned}
f(p) - f(\hat{p}) &= \sum_{i=1}^{M} \sum_{t=1}^{T} \left( a_t^i - s_t^i \right) \left[ p_t^i - \hat{p}_t^i \right] - s_t^i \left[ (p_t^i)^2 - (\hat{p}_t^i)^2 \right] + \sum_{i=1}^{M} \sum_{t=1}^{\min(1,K)} \sum_{k=0}^{M} \alpha^k s_{t-k} \left[ p_t^{(i)} - \hat{p}_t^{(i)} \right] I_{(t,k,i)} \\
&= \sum_{i=1}^{M} \sum_{t=1}^{T} \left( a_t^i - s_t^i \right) p_t^i - \sum_{i=1}^{M} \sum_{t=1}^{T} s_t^i \left( (p_t^i)^2 - (\hat{p}_t^i)^2 \right) + \sum_{i=1}^{M} \sum_{t=1}^{\min(1,K)} \sum_{k=0}^{M} \alpha^k s_{t-k} \left[ p_t^{(i)} - \hat{p}_t^{(i)} \right] I_{(t,k,i)}
\end{aligned}
\]
Since every positive cross product term, $\alpha^k s^i_{t-k} p^{(t)}_i [p(t)_{i,t-k} I(t,k,i)]$, in $f(p)$ will contribute to \( \frac{\partial f}{\partial p_i^{(t)}} = \ldots + \alpha^k s^i_{t-k} p^{(t)}_i p(t)_{i,t-k} I(t,k,i) + \ldots \) from this, we know that each term, $\alpha^k s^i_{t-k} p^{(t)}_i [p(t)_{i,t-k} I(t,k,i)]$, contributes two terms to $\nabla f(p)(p - \bar{p})$ through $\alpha^k s^i_{t-k} p^{(t)}_i I(t,k,i)$ and $\alpha^k s^i_{t-k} p^{(t)}_i [p(t)_{i,t-k} I(t,k,i)]$.

After some algebraic manipulation, we have

\[
\nabla f(p)(p - \bar{p}) = \sum_{i=1}^{M} \sum_{t=1}^{T} (s^i_{t} p^{(t)}_i - \bar{p}_i - s^i_{t} (2\bar{p}_i p^{(t)}_i - 2\bar{p}_i^2)) - \sum_{i=1}^{M} \sum_{t=1}^{T} \sum_{k=0}^{\min(t-1,K)} \alpha^k s^i_{t-k} \left[ 2p^{(t)}_l p^{(t)}_i - 2(\bar{p}_i^2)^2 \right] I(t,k,i)
\]

When we subtract (15) from (14), cancel the appropriate terms and complete the squares, we obtain:

\[
f(p) - f(\bar{p}) - \nabla f(\bar{p})(p - \bar{p}) = \sum_{i=1}^{M} \sum_{t=1}^{T} \left[ s^i_{t} p^{(t)}_i - \bar{p}_i^2 - s^i_{t} (2\bar{p}_i p^{(t)}_i - 2\bar{p}_i^2) \right] - \sum_{i=1}^{M} \sum_{t=1}^{T} \sum_{k=0}^{\min(t-1,K)} \alpha^k s^i_{t-k} \left[ 2p^{(t)}_l p^{(t)}_i - 2(\bar{p}_i^2)^2 \right] I(t,k,i)
\]

We split the first and second terms into two halves as follows:

\[
f(p) - f(\bar{p}) - \nabla f(\bar{p})(p - \bar{p}) = \frac{1}{2} \sum_{i=1}^{M} \sum_{t=1}^{T} \left[ \alpha^0 s^i_{t} \left[ p^{(t)}_i - \bar{p}_i \right]^2 \right] - \sum_{i=1}^{M} \sum_{t=1}^{T} \sum_{k=0}^{\min(t-1,K)} \alpha^k s^i_{t-k} \left[ p^{(t)}_i p^{(t)}_i - \bar{p}_i^2 \right] I(t,k,i)
\]

We complete our proof by showing that each strictly positive term in the last summation is outweighed by two corresponding negative terms from the first three lines.

First consider terms with $k = 0$ in the last line, i.e. $\alpha^0 s^i_{t} \left[ p^{(t)}_i - \bar{p}_i \right]^2 \left( p(t)_{i,t-i} - \bar{p}(t)_{i,t-i} \right) I(t,i)$, Note that $(t,k,i) = i$ when $k = 0$. Therefore, picking $-\frac{1}{2} \alpha^0 s^i_{t} \left[ p^{(t)}_i - \bar{p}_i \right]^2$ and $-\frac{1}{2} \alpha^0 s^i_{t} \left[ p^{(t)}_i - \bar{p}_i \right]^2$ from the first two lines, completing the squares, we obtain: $-\frac{1}{2} \alpha^0 s^i_{t} \left[ p^{(t)}_i - \bar{p}_i \right]^2 I(t,i)$. Now, consider the terms with $k > 0$ in the last line. We pair $\alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right]^2 \left( p(t)_{i,t-k} - \bar{p}(t)_{i,t-k} \right) I(t,k,i)$ from the last line with the corresponding $-\frac{1}{2} \alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right]^2 I(t,k,i)$ from the second line. Since $(t,k,i)$ is the period at which the minimum price is offered between period $t - k$ and $t - 1$, if $t - k < (t,k,i) \leq t - 1$, there must be a period between period $t - k + 1$ and period $t - 1$ such that there exists the residual demand originating from period $t - k$ realized at price $\bar{p}(t)_{i,t-k}$, and thus the corresponding term, $-\frac{1}{2} \alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right]^2 I(t,i)$, where $n = (t,k,i) - (t - k)$ can be selected from the second line. On the other hand, if $(t,k,i) = t - k$, that is $\bar{p}^{(t-k)}_{i,t-k}$ is the minimum price for the next $k$ period, a corresponding term, $-\frac{1}{2} \alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right]^2 I(t,i)$, must exist.

Since $\alpha^k$ are decreasing in $k$, we have $\frac{1}{2} \alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right]^2 \leq -\frac{1}{2} \alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right]^2 \left( p(t)_{i,t-k} - \bar{p}(t)_{i,t-k} \right)$ for $0 \leq j \leq k$. Since every positive cross product term can be matched with two corresponding negative terms, a little algebra shows:

\[
f(p) - f(\bar{p}) - \nabla f(\bar{p})(p - \bar{p}) \leq -\frac{1}{2} \sum_{i=1}^{M} \sum_{t=1}^{T} \sum_{k=0}^{\min(t-1,K)} \alpha^k s^i_{t-k} \left[ p^{(t)}_i - \bar{p}_i \right] \left( p(t)_{i,t-k} - \bar{p}(t)_{i,t-k} \right) I(t,k,i) \leq 0
\]
A.2. Proof of Lemma 1

Suppose that there exists a pricing plan that has multiple products that have the same minimum price at a time period \( t \). Let \( D \) be the realized demand of this pricing plan. We will show that there exists another pricing plan with higher profit. Let \( m \) be the index of product in the list \( j(t) \) that has the lowest unit cost, i.e. \( m = \arg \min_{i \in j(t)} c_i \). If there are more than one product that has the lowest cost, then without loss of generality pick any one of them. Now, we increase prices of products in \( j(t) \setminus \{m\} \) by \( \epsilon \). Note that increasing them by \( \epsilon \) would not change the total demand realization at period \( t \) since demand lost by these products will be gained by the product \( m \). On the other hand, this increases the total profit at period \( t \) by \( \sum_{i \in j(t) \setminus \{m\}} s_i \epsilon \) since demand \( \sum_{i \in j(t) \setminus \{m\}} s_i \) paid \( p_m \) previously and now pays \( p_m + \epsilon \). In order to complete the proof, we need to find a feasible inventory ordering policy and show that this does not increase the cost. In order to do this, we just take the capacities that were originally used to satisfy demand for the products in \( j(t) \setminus \{m\} \) and shift them to product \( m \). Note that by construction \( c_i^* \leq c_i \) for all \( i \in j(t) \setminus \{m\} \) and the assumption assures that shifting capacity to product \( m \) always leads to a plan with lower cost.

A.3. Proof of Lemma 2:

Suppose that Assumption 2 holds and optimal \( j(t) \) is not equal to the index of cheapest product \( i \). Without loss of generality, assume that \( c_1^* \leq c_2^* \leq \ldots \leq c_M^* \) and let \( j(t) = 2 \), i.e. \( p_1^* \leq p_2^* \). Note that replacing price of product 2 with the price of product 1, i.e. \( p_1^* \leftarrow \min(p_1^*, p_2^*) \) and \( p_2^* \leftarrow \max(p_1^*, p_2^*) \) will not change the total residual and substitute demands. Therefore, the exchange will always increase the total profit since the change in total profit generated by change in demands \( D_1^*(\min(p_1^*, p_2^*)) - D_1^*(\min(p_1^*, p_2^*)) > 0 \) and margins \( c_1^* - c_2^* > 0 \) is positive due to the Assumption 2.

A.4. Proof of Lemma 3:

We assume that \( q = \infty \), and \( \beta = 1 \). With these assumptions, the profit function can be written as follows:

\[
\pi(p_i) = \sum_{i=1}^{M} (p_i - c_i) \left[ a^i - s^i p_i + \sum_{j=1}^{M} s^j [p_i^j - p_i^j]_+ + \sum_{j=1}^{M} \alpha_s^j \left[ \min_{l=1,M} p_i^l - p_i^1 \right]_+ \right]
\]

Let \( i^* \) be the index for the cheapest product. Then, profit function can be rewritten as follows:

\[
\pi(p_1^i, \ldots, p_M^i) = (p_i^* - c_i^*) \left( a^{i^*} - s^{i^*} p_i^* + \sum_{i \neq i^*, i=1}^{M} s^i (p_i^i - p_i^{i^*}) + \sum_{i=1}^{M} \alpha_s^i (p_i^{i^*} - p_i^{i^*}) \right) + \sum_{i \neq i^*, i=1}^{M} (p_i^i - c_i^*)(a^i - s^i p_i^i)
\]

Differentiating the above function with respect to \( p_i^i \), where \( i \neq i^* \), we obtain the following first order condition:

\[
\frac{\partial \pi(p_1^i, \ldots, p_M^i)}{\partial p_i^i} = a^i - 2s^i p_i^i + s^i c_i^i + s^i (p_i^{i^*} - c_i^{i^*})
\]

Setting it equal to 0, and solving for \( p_i^i \), we obtain:

\[
p_i^i = \frac{a^i + s^i c_i^i + s^i (p_i^{i^*} - c_i^{i^*})}{2s^i}
\]
A.5. Proof of Proposition 3

Consider a $N$-period non-increasing price sequence $p$ and without loss of generality, assume that $j(t) = 1$ for all $t \in T$, i.e. $p_1 \geq p_2 \geq \ldots \geq p_T$ for all $i \in M$, and $p_t \leq p_i$ for all $t \in T$ and $i \in M$. Then profit function can be written as follows:

$$f_N(p) = \sum_{i=1}^{N} (p_i - c^i)(a^i - s^i p_i) + \sum_{i=1}^{N} (p_i - c^i)s^i(p_i - p_{i-1})$$

$$f_N(p) = \sum_{i=1}^{N} (p_i - c^i)(a^i - s^i p_i) + \sum_{i=1}^{N} \beta(p_i - c^i)(p_i - p_i) + \sum_{i=1}^{N} \alpha(p_i - c^i)(p_i - p_i)$$

Let $\hat{p}^i$ denote the non-interaction optimal for product $i$. Then, by letting $p_i = \hat{p}^i(1 + \delta^i)$, substituting $a^i + c^i s^i = 2s^i \hat{p}^i$, and subtracting the resulting expression from $f_N(\hat{p})$, we obtain:

$$f_N(p) - f_N(\hat{p}) = \sum_{i=1}^{N} -s^i (\hat{p}^i)^2(\delta^i)^2 + \sum_{i=1}^{N} [s^i \hat{p}^i \delta^i \delta^i - s^i (\hat{p}^i)^2 (\delta^i)^2] + \sum_{i=1}^{N} [\alpha_s (\hat{p}^i)^2 \delta^i \delta^i - \alpha_s (\hat{p}^i)^2 (\delta^i)^2]$$

$$f_N(p) - f_N(\hat{p}) = \sum_{i=1}^{N} [s^i \hat{p}^i \delta^i \delta^i + \beta (\hat{p}^i)^2 (\delta^i)^2] + \sum_{i=1}^{N} [\alpha (\hat{p}^i)^2 \delta^i \delta^i - \alpha (\hat{p}^i)^2 (\delta^i)^2]$$

Dividing by the sequence length, $N$, we get the average profit per period $\Delta f_N$, when $N$-period pricing sequence is used. In matrix form, we can rewrite $\Delta f_N$ as follows:

$$\Delta f_N = \sum_{i=1}^{N} \left[ \frac{\hat{p}^i}{N} \right] \left[ \frac{1}{2} \alpha \delta^i \delta^i - \epsilon \delta^i \cdot \delta^i \right]$$

where $\Omega$ is $MN \times MN$ matrix, and $e$ is $MN \times 1$ vector. Let $\Omega_{ij}$ be $N \times N$ dimensional sub-matrix in $\Omega$, where $i,j = 1 \ldots M$. $\Omega_{ij}$ can be expressed as follows: $\Omega_{ii} = \alpha_{ii} = \frac{1}{\epsilon_i} I_{N \times N}$, $\Omega_{ij} = -2s^i (\hat{p}^j / \hat{p}^i)^2 I_{N \times N}$, where $I_{N \times N}$ is $N \times N$ dimensional unit matrix and $s^i = s^i / \sum_{j=1}^{M} s^j$. Likewise, $e_i$ can be expressed as follows: $e_i = m \sum_{j=1}^{M} s^j (1 - r^j) I_{N \times 1} - m \alpha J_{N \times 1}$ and $e_i = -m \sum_{j=1}^{M} s^j \hat{p}^j I_{N \times 1}$, where $J_{N \times 1} = (1, \ldots, 1)^T$, $J_{N \times 1} = (-1, 0, \ldots, 0, 1)^T$, $m = 1 - c^i / \hat{p}^i$, and $r^j = \frac{c^j - c^i}{2 \hat{p}^i \hat{p}^j}$. We can take the derivative of quadratic form and set it to zero to obtain the optimal $\delta$ and $\gamma$ in matrix notation:

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1M} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M1} & \Omega_{M2} & \cdots & \Omega_{MM} \end{pmatrix} \begin{pmatrix} \delta^1 \\ \delta^2 \\ \vdots \\ \delta^M \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_M \end{pmatrix}$$

Solving $\delta^i$ in terms of $\delta^i$ in (17), we obtain: $\delta^i = -\Omega_{ii}^{-1} \cdot \Omega_{1i} \cdot \delta^1 + \Omega_{ii}^{-1} \cdot e_i$ $\rightarrow \delta^i = \frac{\delta^i}{2r^i} + \frac{\delta^i}{2r^i}$. Substituting it back to (17), we obtain the following equation for $\delta^i$: $\left( \Omega_{ii} + \sum_{j=2}^{M} \Omega_{ii} \cdot \frac{\hat{p}^j}{2r^j} \right) \cdot \delta^i = \left( e_i - \sum_{j=2}^{M} \Omega_{ii} \cdot \frac{\hat{p}^j}{2r^j} \right)$. Dividing both sides by $\alpha$, we obtain the following matrix equation:

$$\begin{pmatrix} -1 & \frac{1}{1 - u - 2} & \frac{1}{1 - u - 2} & \frac{1}{1 - u - 2} & \frac{1}{1 - u - 2} \\ \frac{1}{1 - u - 2} & 0 & 0 & 0 & 0 \\ \frac{1}{1 - u - 2} & 0 & 0 & 0 & 0 \\ \frac{1}{1 - u - 2} & 0 & 0 & 0 & 0 \\ \frac{1}{1 - u - 2} & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \delta = \begin{pmatrix} 1 - \frac{1}{1} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} z m$$
where \( \boldsymbol{\delta} = (\delta_1, \ldots, \delta_N)^T, \) \( u = 2/\alpha - \sum_{i=2}^{M} \tilde{s}_i/(2\alpha) = \frac{1}{2\alpha} [3 + \bar{\tilde{s}}] \) and \( z = \sum_{i=2}^{M} \tilde{s}_i/(2\alpha) - \sum_{i=2}^{M} \tilde{s}_i r'/\alpha = \frac{1}{\alpha} \sum_{i=2}^{M} \tilde{s}_i \left[ \frac{1}{2} - r' \right] \). We can rewrite the right hand side as follows:

\[
\begin{pmatrix}
1 - \frac{1}{2} & 1 - \frac{1}{2} \\
1 & 1 \\
\end{pmatrix} \begin{pmatrix}
p^2 \\
0 \\
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2} \\
\frac{1}{2} \\
\end{pmatrix} \begin{pmatrix}
z m \\
\end{pmatrix} + \frac{1}{\alpha} \sum_{i=2}^{M} \tilde{s}_i \left[ \frac{1}{2} - r' \right].
\]

In Lemma 5, we find \( \gamma \) and \( \zeta \) such that \( \Omega \gamma m = \mathbf{p}^i \) and \( \Omega \zeta m = \mathbf{p}^2 \). Hence, \( \delta^1 = (\gamma + \zeta) m \) satisfies (18).

Hence, \( \delta^1 = (\gamma_0 - \frac{\bar{\tilde{s}}}{u}) m \) is the optimal perturbation. Note that we can rewrite \( \gamma_0 = \gamma_0 \theta_0 = \theta_0 = 1 \). This implies that optimal price of product 1 is \( p_1^* = \bar{\tilde{p}}^1 (1 + \delta_1) \) where \( \theta_0 = \frac{\beta_{N-2} \theta_0 - 1}{\beta_{N-1} - 1} \) for \( t = 1 \ldots N, \theta_0 = 1 \). This implies that optimal price for product 1 is \( p_1^* = \bar{\tilde{p}}^1 (1 + \delta_1) \) where \( \kappa_t = \frac{1 + \bar{\tilde{s}}}{\beta_{N-t} - 1} \). Recall that when \( i > 1, \delta_i = \frac{\kappa_i \tilde{s}_i}{\beta_{N-i} - 1} \), therefore this implies that optimal price for product \( i > 1 \) is \( p_i^* = \bar{\tilde{p}}_i (1 + \delta_i) = \bar{\tilde{p}}_i + \frac{\kappa_i \tilde{s}_i}{\beta_{N-i} - 1} \left[ (\tilde{p}_i - 1) (1 + \tilde{s}_i) + \sum_{j=2}^{M} \tilde{s}_j (\tilde{p}_j - \tilde{p}_i) \right] \)

**LEMMA 5.** (i) Let \( \gamma = [\gamma_1 \gamma_2 \ldots \gamma_{N-1} \gamma_N]^T \) where \( \gamma_i = \frac{\beta_{N-i-1} - \gamma_0}{\beta_{N-i} - 1} \) for \( t = 1 \ldots N, \gamma_0 = (1 - \frac{\bar{\tilde{s}}}{u}), \frac{\bar{\tilde{s}}}{u} = \sum_{i=2}^{M} s_i [\bar{\tilde{s}} - r'] \) and \( \beta_i = 1, \beta_i = u + 2, \) and \( \beta_i = -u \beta_{i-1} - \beta_{i-2} \) for \( t = 2 \ldots N - 1, \beta_N = u \).

Then, \( \gamma \) satisfies \( \Omega \gamma m = \mathbf{p}^i \)

(ii) \( \zeta = [-\frac{\bar{\tilde{s}}}{u} - \frac{\bar{\tilde{s}}}{u} - \frac{\bar{\tilde{s}}}{u} - \frac{\bar{\tilde{s}}}{u}]^T \) satisfies \( \Omega \zeta m = \mathbf{p}^2 \)

It is easy to prove this result by just substituting the given \( \gamma \) and \( \zeta \) in to the equations.

**A.6. Proof of Proposition 4**

If we substitute the optimal \( \delta = \Omega^{-1} e \) into quadratic form (16) to obtain the optimal average profit per period, \( \Delta f_N^* \):

\[
\Delta f_N^* = \frac{\sum_{i=1}^{M} s_i (\tilde{p}_i)^2}{N} \left[ \frac{1}{2} e^{T} \Omega^{-1} e \right] = \frac{\sum_{i=1}^{M} s_i (\tilde{p}_i)^2}{N} \left[ \frac{1}{2} e^{T} \Omega^{-1} e \right] = \frac{\sum_{i=1}^{M} s_i (\tilde{p}_i)^2}{2N} \sum_{i=1}^{M} (e_i^T \delta^i)
\]

Substituting \( \delta^i = -\Omega_i^{-1} \Omega_i \delta^1 + \Omega_i^{-1} e_i \):

\[
\Delta f_N^* = -\frac{\sum_{i=1}^{M} s_i (\tilde{p}_i)^2}{2N} \left[ e_i^T \delta^i - \sum_{i=2}^{M} e_i^T \Omega_i^{-1} \Omega_i \delta^1 + \sum_{i=2}^{M} e_i^T \Omega_i^{-1} e_i \right] = -\frac{\sum_{i=1}^{M} s_i (\tilde{p}_i)^2}{2N} \left[ e_i^T - \sum_{i=2}^{M} e_i^T \Omega_i^{-1} \Omega_i \right] \delta^1 + \sum_{i=2}^{M} e_i^T \Omega_i^{-1} e_i \]

where

\[
\left( e_1^T - \sum_{i=2}^{M} e_i^T \Omega_i^{-1} \Omega_i \right) = m \left( \begin{pmatrix} \sum_{i=2}^{M} s_i (1/2 - r') \\ \vdots \\ \sum_{i=2}^{M} s_i (1/2 - r') \end{pmatrix} \right) + \left( \begin{pmatrix} -\frac{\bar{\tilde{s}}}{u} \\ \vdots \\ -\frac{\bar{\tilde{s}}}{u} \end{pmatrix} \right) \sum_{i=2}^{M} s_i \Omega_i^{-1} e_i = -\frac{m^2 N}{2} \sum_{i=2}^{M} s_i \Omega_i^{-1} e_i
\]
Substituting these terms back into (22),
\[
\Delta f_N^* = -\frac{\sum_{i=1}^M s^i (\hat{p}^i)^2}{2N} \left[ m \left( \frac{1}{u} \sum_{i=2}^M s^i (1/2 - r^i) - \alpha \right) (\delta_1^i - \delta_N^i) - \frac{m^2Nz}{2} \sum_{i=2}^M \bar{s}^i \right]
\]
Note that \( \sum_{i=1}^N \delta_i^i = \frac{1}{2} (\delta_1^i - \delta_N^i) - N \frac{mz}{u} \) due to the the row sum of matrix equation. Therefore, \( \Delta f_N^* \) can be expressed as follows:
\[
\Delta f_N^* = -\frac{\sum_{i=1}^M s^i (\hat{p}^i)^2}{2N} \left[ m \left( \frac{1}{u} \sum_{i=2}^M s^i (1/2 - r^i) - \alpha \right) (\delta_1^i - \delta_N^i) - \frac{m^2Nz}{2} \sum_{i=2}^M \bar{s}^i \right]
\]
Recall that \( \delta_1^i - \delta_N^i = m (\gamma_1 - \gamma_N) \). Substituting it into \( \Delta f_N^* \) and collecting terms that depend on \( N \), we obtain:
\[
\Delta f_N^* = \frac{\sum_{i=1}^M s^i (\hat{p}^i)^2}{2} \left[ m^2 \left( \alpha - \frac{1}{u} \sum_{i=2}^M s^i (1/2 - r^i) \right) \left( \gamma_1 - \gamma_N \right) / N + \frac{m^2z}{2} \sum_{i=2}^M \bar{s}^i + \frac{m^2z}{u} \sum_{i=2}^M \bar{s}^i (1/2 - r^i) \right]
\]
\begin{equation}
A_1(\gamma_1 - \gamma_N)/N + A_2
\end{equation}
where \( A_1 = \frac{\alpha (\hat{p}^1)^2 \Sigma^M_{i=1} s^i}{2(3 + \bar{s}^i)} \) and \( A_2 = \frac{\alpha (\hat{p}^i)^2 \Sigma^M_{i=1} s^i}{2(3 + \bar{s}^i)} \). Note that the increase in the average profit under an optimal non-increasing price sequence of length \( N \) depends only on the difference between the price distortion of product 1 (from its noninteraction price \( \hat{p}^i \)) in the first period (i.e., \( \gamma_1^i \)) and the price distortion in the last period (i.e., \( \gamma_N^i \)). Both quantities depend on the length of a price sequence, \( N \). In order to show that the price sequence of length 2 is optimal, we develop the upper bound for the average profit increase when a \( N \)-period optimal decreasing pricing sequence is used. Using Lemma 8, we characterize asymptotic properties for both \( \gamma_1 \) and \( \gamma_N \), which will be useful when we prove the optimality of period 2. For any optimal decreasing pricing sequence of length \( N \), the average profit improvement is bounded by:
\[
\Delta f_N^* = A_1(\gamma_1 - \gamma_N)/N + A_2 \leq \Delta F_N^* = \frac{A_1}{N} \left[ \frac{1}{r_1 - r_{\infty}} + \frac{1}{r - 1} \right] + A_2
\]
It is trivial to show that \( \Delta F_N^* \) decreases in \( N \), therefore, following relational structure holds:
\[
\begin{align*}
\Delta f_2^* &\leq \Delta f_N^* \\
A_1(\gamma_1 - \gamma_2)/2 + A_2 &\leq \frac{A_1}{N} \left[ \frac{1}{r_1 - r_{\infty}} + \frac{1}{r - 1} \right] + A_2
\end{align*}
\]
\[
\Delta f_3^* \leq \Delta f_N^* \quad A_1(\gamma_1 - \gamma_3)/3 + A_2 \leq \frac{A_1}{N} \left[ \frac{1}{r_1 - r_{\infty}} + \frac{1}{r - 1} \right] + A_2
\]
\[
\vdots \quad \vdots \quad \Rightarrow \quad \vdots
\]
\[
\Delta f_n^* \leq \Delta f_N^* \quad A_1(\gamma_1 - \gamma_N)/N + A_2 \leq \frac{A_1}{N} \left[ \frac{1}{r_1 - r_{\infty}} + \frac{1}{r - 1} \right] + A_2
\]
If we can show that \( A_1(\gamma_1 - \gamma_2)/2 + A_2 \geq \frac{A_1}{N} \left[ \frac{1}{r_1 - r_{\infty}} + \frac{1}{r - 1} \right] + A_2 \) for some \( N \) then the above relational diagram implies that \( \Delta f_2^* \geq \Delta f_n^* \) where \( n \geq N \). After some algebra, we can show that for all \( s^i \in [0,1] \) and \( \alpha \in (0,1] \):
\[
\frac{\gamma_1 - \gamma_2}{2} = 2\alpha(3 + s^1) \geq \frac{2\alpha(3 + s^1)}{(3 + s^1)(3 + s^1 + 4\alpha) - 4\alpha^2} \geq \gamma_0 \left( \frac{1}{4} \left( \frac{1}{r_1 - r_{\infty}} + \frac{1}{r - 1} \right) \right) = \gamma_0 \left( \frac{2}{u - 2 + \sqrt{u^2 + 4u} + 1} + \frac{1}{u} \right)
\]
\[
= \gamma_0 \left[ \frac{2\alpha}{4} \left( \frac{3(3 + s^1) + \sqrt{(3 + s^1)^2 + 8\alpha(3 + s^1)}(3 + s^1 + 2\alpha)}{(3 + s^1 - 4\alpha + \sqrt{(3 + s^1)^2 + 8\alpha(3 + s^1)})(3 + s^1 + 2\alpha)} \right) \right]
\]
Hence \( \Delta f_3^* \geq \Delta F_3^* \geq \Delta f_3^* \) where \( n \geq 4 \). There remains to prove that \( \Delta f_3^* \geq \Delta f_3^* \). In fact, we can easily show that for all \( s^1 \in [0,1] \) and \( \alpha \in [0,1] \):

\[
\gamma_1 - \gamma_2 = \frac{\theta_1 - \theta_2}{2} = 2\alpha(3 + s^1) - 4\sqrt{\alpha(3 + s^1)} \geq \gamma_0 \frac{\theta_1 - \theta_2}{3} = \gamma_0 \frac{(\gamma_0 - 3 + s^1) + 6\alpha}{3 + s^1 + 8\alpha + 8\alpha^2(3 + s^1 - 2\alpha)}
\]

which implies that \( \Delta f_3^* \geq \Delta f_3^* \). This completes the proof.

Lemma 6 and 7 are used in the proof of Lemma 8. Due to the lack of space, we provide them without proof:

**Lemma 6.** Let \( r = 2 + u \) For all \( n \geq 1 \), \( \beta_{n-1}(n)(\beta_n(n) \geq r^i \), \( i = 1, ..., n \)

**Lemma 7.** \( \lim_{n \to \infty} \frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{1}{r_1 - r_{\infty}} \) where \( r_{\infty} = \frac{u + 2 - \sqrt{u^2 + 4u}}{2} \) and \( r_1 = u \).

**Lemma 8.** (i) \( \gamma_1(n) \) is increasing in \( n \) and \( \lim_{n \to \infty} \gamma_1(n) = \frac{\gamma_0}{r_1 - r_{\infty}} \).

(ii) \( \liminf_{n \to \infty} \gamma_n \geq \frac{r_1 - r_{\infty}}{2} \) for all \( n \geq 1 \).

where \( r_{\infty} = \frac{u + 2 - \sqrt{u^2 + 4u}}{2} \), \( r_1 = u \) and \( r = 2 + u \).

Proof of Lemma 8. From Lemma 7:

\[
\gamma_1(n) = \frac{\beta_{n-1}(n)\gamma_0 - \gamma_0}{\beta_n(n)}.
\]

It can be easily shown that \( \gamma_1(n) \) increases as \( n \) increases by using the fact that both \( \frac{\beta_{n-1}(n)}{\beta_n(n)} \) and \( \beta_n(n) \) are monotone increasing. Furthermore, the convergence of a sequence, \( \frac{\beta_{n-1}(n)}{\beta_n(n)} \) and the fact that \( \lim_{n \to \infty} \beta_n(n) = \infty \) imply the existence of the limit. Replacing \( \gamma_0 \) with \( a \) as in Lemma 7, we have: \( \lim_{n \to \infty} \gamma_1(n) = \lim_{n \to \infty} \frac{\beta_{n-1}(n)}{\beta_n(n)} = \frac{\gamma_0}{r_1 - r_{\infty}} \).

To show the inequality, we consider the following system of difference equations: \( \gamma_1(n) = \frac{\beta_{n-1}(n)}{\beta_n(n)} \gamma_0 - \frac{\beta_n(n)}{\beta_n(n)} \gamma_0 \), \( \gamma_2(n) = \frac{\beta_{n-2}(n)}{\beta_{n-1}(n)} \gamma_0 - \frac{\beta_{n-1}(n)}{\beta_{n-1}(n)} \gamma_0 \), ..., \( \gamma_n(n) = \frac{\beta_n(n)}{\beta_n(n)} \gamma_0 - \gamma_0 \left( \frac{\beta_{n-1}(n)}{\beta_{n-1}(n)} \beta_{n-1}(n) + \frac{\beta_{n-2}(n)}{\beta_{n-2}(n)} \beta_{n-2}(n) + \cdots + \frac{\beta_0(n)}{\beta_0(n)} \beta_0(n) \right) \). It is very easy to see that \( \gamma_n(n) \) is decreasing in \( n \) since the positive term in the above expression decreases while more negative term is added as \( n \) increases. To get a bound of \( \gamma_n(n) \), we take a close look at \( \beta_n(n) \beta_{n-1}(n) \) terms. Lemma 6 implies that \( \beta_n(n) \beta_{n-1}(n) \) is bounded below by \( r^i \) and substituting the corresponding terms yields:

\[
\gamma_n(n) = \frac{\beta_0(n)}{\beta_n(n)} \gamma_0 - \frac{\beta_0(n)}{\beta_n(n)} \left( \frac{\beta_0(n)}{\beta_0(n)} + r^{n-1} \frac{\beta_0(n)}{\beta_0(n)} + \cdots + r^2 \frac{\beta_0(n)}{\beta_0(n)} + r \frac{\beta_0(n)}{\beta_0(n)} \right) - \gamma_0 \left( \frac{\beta_0(n)}{\beta_0(n)} + \frac{\beta_0(n)}{\beta_0(n)} + \cdots + \frac{\beta_0(n)}{\beta_0(n)} \right) \gamma_0.
\]

Second last inequality holds because \( \frac{\beta_0(n)}{\beta_0(n)} \gamma_0 \) is nonnegative and the last equality comes from \( \beta_0(n) = 1 \).

A.7. Proof of Proposition 5

We proved in Proposition 4 that the optimal revenue increase is generated by repeating the 2-period optimal pricing sequence, which implies that if the planning period \( T \) is an even number then the pricing plan is simply repeating the 2-period optimal pricing sequence \( \frac{T}{2} \) times. However, when \( T \) is odd, a small modification of the pricing strategy is required for the last three periods. Note that there are two ways to construct a 3-period pricing plan. First way is to use \( \langle p_{01}, p_{02} \rangle \) for the first two periods, solve a single period profit maximization

\footnote{Proofs of Lemma 6 and 7 are available from the author upon request.}
problem that takes into account only substitution effect and use the optimal single period prices \( p_{\text{single}} \) for the last period. Second way is to use 3-period decreasing pricing sequence \( (p_{3,\text{hi}}, p_{3,\text{me}}, p_{3,\text{lo}}) \) for the last three periods, where

\[
p_i^t = \begin{cases} 
  c_i + \frac{2\alpha}{2s_i} \left( (\hat{p}^i - c_i)(\hat{s}^i + 1) + \sum_{j=2}^{M} \hat{s}_j (\hat{p}^j - \hat{p}^i) \right) & \text{for } i = 1 \\
  \hat{p}_i + \frac{2\alpha}{2s_i} \left( (\hat{p}^i - c_i)(\hat{s}^i + 1) + \sum_{j=2}^{M} \hat{s}_j (\hat{p}^j - \hat{p}^i) \right) & \text{for } i > 1.
\end{cases}
\]

for all \( t \in \{3 - hi, 3 - med, 3 - lo\} \) and \( R_{3,\text{hi}} = (s^i + 3 + 6\alpha)(s^i + 3 + 4\alpha) \), \( R_{3,\text{me}} = (s^i + 3 + 8\alpha)(s^i + 3) + 8\alpha^2 \), \( R_{3,\text{lo}} = (s^i + 3)(s^i + 3 + 6\alpha) \), \( R = (s^i + 3)^2(s^i + 3 + 8\alpha) + 8\alpha(s^i + 3 - 2\alpha) \).

The optimal pricing plan for the last three periods is determined by whichever of two strategies generate the most profit. First, we calculate the profit of \( (p_{hi}^t, p_{lo}^t, p_{single}) \). Recall that difference between base profit and profit generated by \( (p_{hi}^t, p_{lo}^t) \) is equal to \( 2\Delta f_2 = A_1(\gamma_1 - \gamma_2) + 2A_2 \). To calculate the profit at the last period, we need to compute first \( p_{single} \). We will only provide the closed form expression for \( p_{single} \):

\[
p_{\text{single}}^t = \begin{cases} 
  c_i + \frac{2\alpha}{2s_i} \left( (\hat{p}^i - c_i)(\hat{s}^i + 1) + \sum_{j=2}^{M} \hat{s}_j (\hat{p}^j - \hat{p}^i) \right) & \text{for } i = 1 \\
  \hat{p}_i + \frac{2\alpha}{2s_i} \left( (\hat{p}^i - c_i)(\hat{s}^i + 1) + \sum_{j=2}^{M} \hat{s}_j (\hat{p}^j - \hat{p}^i) \right) & \text{for } i > 1.
\end{cases}
\]

The difference between base profit and profit generated by \( p_{single} \), \( \Delta f_1 \) is

\[
\Delta f_1 = \frac{(\hat{p}^i - c_i)^2}{3 + \hat{s}_1} \sum_{i=2}^{M} \hat{s}_i \left( \frac{1}{4}(1 - \hat{s}_1)(3 + \hat{s}_1) + \sum_{i=2}^{M} \hat{s}_i \left( \frac{1}{2} - \frac{\hat{p}_i - \hat{p}^i}{\hat{p}^i - c_i} \right) \right)^2.
\]

Note that \( \Delta f_1 = A_2 \). Hence total increase in the profit generated by \( (p_{hi}^t, p_{lo}^t, p_{single}) \) is equal to \( A_1(\gamma_1 - \gamma_2) + 3A_2 \).

Second way of constructing 3-period is \( (p_{3,\text{hi}}, p_{3,\text{me}}, p_{3,\text{lo}}) \), which leads to a total increase in profit by \( A_1(\gamma_1 - \gamma_3) + 3A_2 \). Therefore, it is enough to show that \( A_1(\gamma_1(2) - \gamma_2(2)) + 3A_2 \leq A_1(\gamma_1(3) - \gamma_3(3)) + 3A_2 \). But it is true, since \( \gamma_1(n) \) increases in \( n \) and \( \gamma_i(n) \) decreases in \( n \) by Lemma 8.

A.8. Proof of Proposition 6

In Propositions 1 and 4, we obtain closed form expressions for optimal one-period high - one period high pricing strategy for model \( (P_s) \) when \( \beta = 0 \) and \( \beta = 1 \), respectively. Using these expressions, we can express the absolute difference between high and low prices for products \( i^* \) and \( i \) as follows:

\[
\Delta^A(\beta = 0) = \begin{cases} 
  \left( \hat{p}^{i^*} - c^{i^*} \right) \frac{4\alpha}{4\alpha + 4 - \alpha^2} & \text{if } i = i^* \\
  \left( \hat{p}^i - c^i \right) \frac{4\alpha}{4\alpha + 4 - \alpha^2} & \text{if } i \neq i^*
\end{cases}
\]

and

\[
\Delta^A(\beta = 1) = \begin{cases} 
  \left( \frac{8\alpha}{(i^* + 3)(4\alpha + 4 - \alpha^2)} \left[ (\hat{p}^{i^*} - c^{i^*})(\hat{s}^{i^*} + 1) + (\hat{p}^i - \hat{p}^{i^*})(\hat{s}^i) \right] \right) & \text{if } i = i^* \\
  \left( \frac{8\alpha}{(i^* + 3)(4\alpha + 4 - \alpha^2)} \left[ (\hat{p}^{i^*} - c^{i^*})(\hat{s}^{i^*} + 1) + (\hat{p}^i - \hat{p}^{i^*})(\hat{s}^i) \right] \right) & \text{if } i \neq i^*
\end{cases}
\]

where \( \hat{p}^i = \frac{a_i}{2s_i} + \frac{c_i}{2} \) and \( \hat{p}^{i^*} = \frac{a_i}{2s_i} + \frac{c_i}{2} \). Since \( 8\alpha \geq 4\alpha \), it is trivial to show that

\[
\Delta^A(\beta = 1) \geq \Delta^A(\beta = 1)
\]

We can rewrite \( \hat{p}^i - c^i \) as follows:

\[
\hat{p}^i - c^i = \frac{a^i}{2s^i} + \frac{c^i}{2} = \frac{a^i - s^ic^i}{2s^i} - \frac{1}{2} c^i \left( \frac{a^i}{s^ic^i} - 1 \right)
\]
Following the assumptions for $c^* \leq c^i$ and $\frac{a_{i}^{i*}}{\sigma_{i}^2} \leq \frac{a_{i}^{i}}{\sigma_{i}^2}$, we can show that

$$\hat{p}^i - c^i \leq \bar{p}^i - c^i$$

Hence, the above inequality implies that

$$\Delta^A_i(\beta = 0) \leq \Delta^A_i(\beta = 0)$$

Now, we compare the relative difference between high and low prices for products $i^*$ and $i$. Relative difference for product $i$ can be written as follows:

$$\Delta^R_i(\beta = 0) = \frac{(\hat{p}^i - c^i)}{\bar{p}^i + (\hat{p}^i - c^i)} = \frac{\hat{p}^i - c^i}{\bar{p}^i} = \frac{\frac{a_{i}^{i*} - 1}{\sigma_{i}^2 + 1} \frac{a_{i}^{i}}{4} + \frac{4}{a_{i}^{i} + 4 - a_{i}^{i}}}{}$$

Note that

$$\frac{a_{i}^{i*}}{\sigma_{i}^2} - 1 \leq \frac{a_{i}^{i}}{\sigma_{i}^2} + 1$$

due to the assumption that $\frac{a_{i}^{i*}}{\sigma_{i}^2} \leq \frac{a_{i}^{i}}{\sigma_{i}^2}$. Also, using the fact that $\frac{4 + 4 - a_{i}^{i}}{a_{i}^{i} + 2} > 1$ for all $0 \leq \alpha \leq 1$, we can show that $\Delta^R_i(\beta = 0) \leq \Delta^R_i(\beta = 0)$. Lastly, we show that $\Delta^R_i(\beta = 1) \geq \Delta^R_i(\beta = 1)$. Note that $\Delta^R_i(\beta = 1)$, and $\Delta^R_i(\beta = 1)$ can written as follows:

$$\Delta^R_i(\beta = 1) = \frac{\Delta^A_i(\beta = 1)}{p_{hi}^i} \quad \text{and} \quad \Delta^R_i(\beta = 1) = \frac{\Delta^A_i(\beta = 1)}{p_{hi}^i}$$

Using the fact that $p_{hi}^i \leq p_{hi}^i$, and $\Delta^A_i(\beta = 1) \geq \Delta^A_i(\beta = 1)$, we can show that $\Delta^A_i(\beta = 1) \geq \Delta^A_i(\beta = 1)$.

A.9. Proof of Proposition 7:

We consider each case separately:

1. \(\beta = 0\) case: First of all, we show that \textit{alternating} one-period high and one period low pricing strategy generates more profit than \textit{simultaneous} pricing strategy. Note that products are symmetric (i.e., demand and cost parameters are same) and cost and demand parameters are stationary (i.e., they do not depend on time). This implies that if simultaneous pricing strategy were adopted, high and low prices would be offered at the same time. This would lead to low demand when high prices are offered, and high demand when low prices are offered. If the capacity at each period, i.e., $q$, is less than the level of the high demand, some inventory needs to be carried from the low demand period to high demand period. This increases the cost by the amount that is proportional to the level of inventory, i.e., $h \times$ inventory. However, we can avoid this inventory cost completely by offering high and low prices in alternating fashion. Then, the demand in each period will be constant hence eliminate completely the need for the transfer of additional inventory between periods.

Using the fact that \(I = 0, \beta = 0\), and the stationary and symmetric setting of the problem, we can decompose the problem 9 into independent sub-problems for each product:

$$\max \{ f(p_{hi}^i, p_{lo}^i) \ s.t. \ d_{hi}^i(p_{hi}^i, p_{lo}^i) + d_{lo}^i(p_{hi}^i, p_{lo}^i) \leq q \}$$

where $d_{hi}^i(p_{hi}^i, p_{lo}^i) = a - sp_{hi}^i$, $d_{lo}^i(p_{hi}^i, p_{lo}^i) = (a - sp_{hi}^i) + \alpha(p_{hi}^i - p_{lo}^i)$, and $f(p_{hi}^i, p_{lo}^i) = (p_{hi}^i - c)d_{hi}^i(p_{hi}^i, p_{lo}^i) + (p_{lo}^i - c)d_{lo}^i(p_{hi}^i, p_{lo}^i)$. Let $\lambda \geq 0$ be the lagrangian variable for constraint $d_{hi}^i(p_{hi}^i, p_{lo}^i) + d_{lo}^i(p_{hi}^i, p_{lo}^i) \leq q$. Let $\mathbf{p}^i = (p_{hi}^i, p_{lo}^i)$.
Since the above model is a concave optimization problem with linear constraints, optimal solution \( (p') \) has to satisfy the following KKT conditions: 
\[
\{ \nabla f(p') = \lambda (\nabla d_{hi}(p') + \nabla d_{lo}(p')) ,
\quad d_{hi}(p') + d_{lo}(p') \leq q \}
\]
where \( \lambda \) represents the Lagrange multiplier. Depending on whether \( \lambda = 0 \) or \( \lambda > 0 \), we have 2 cases:
- **Case-1:** \( \lambda = 0 \)

Note that in this case, first KKT condition transform into \( \nabla f(p') = 0 \). The following prices satisfy this condition
\[
p^i_t = \frac{a}{s} - \frac{1}{4s} \left[ \frac{4(a - sc)(1 + \alpha)}{4 + 4a - \alpha^2} \right] \begin{cases} 
2 - \alpha & \text{for } t = hi \\
\frac{2 + 3a - \alpha^2}{4 + 4a - \alpha^2} & \text{for } t = lo. 
\end{cases} 
\tag{26}
\]
In order to meet capacity constraint in the each period, i.e., \( d_{hi}(p') + d_{lo}(p') \leq q \), \( q \) must satisfy \( q \geq \frac{4(a - sc)(1 + \alpha)}{4 + 4a - \alpha^2} \). Therefore, as long as \( q \geq \bar{q} \), the prices in Eqs 26 satisfy all KKT conditions, hence are optimal solution.

- **Case-2:** \( \lambda > 0 \)

In this case, KKT conditions consist of 
\[
\{ \nabla f(p) = \lambda (\nabla d_{hi}(p) + \nabla d_{lo}(p)),
\quad d_{hi}(p) + d_{lo}(p) = q, p^i_{hi} \geq p^i_{lo} \}
\]
The prices that satisfy these equations are as follows:
\[
p^i_t = \frac{a}{s} - \frac{1}{4s} q \left[ \frac{2 - \alpha}{1 + \alpha} \right] \begin{cases} 
\text{for } t = hi \\
\text{for } t = lo. 
\end{cases} 
\tag{27}
\]
Note that \( p^i_{hi} \geq p^i_{lo} \). It is easy to check that as long as \( q < \bar{q}, \lambda > 0 \). Therefore, as long as \( q < \bar{q} \), the prices in Eqs 27 satisfy all KKT conditions, hence are optimal solution.

2. **\( \beta = 1 \)** case: Before we start the analysis, we show that when \( \beta = 1 \), simultaneous one period high and one period low pricing policy leads to higher profit than the alternating one. First, note that Lemma 2 and Lemma 3 still hold true for this special case, which essentially implies that if \( c' \leq c^* \) then in the optimal solution \( p^i_t \leq p^i_t \) for all \( t \) and the optimal price of the product with higher cost can be expressed in terms of the optimal price of the product with lower cost. Since we assume that \( c' = c^* = c \), without loss of generality, we can pick any product and charge it with the lowest price throughout the planning horizon. Let \( i = 1 \) be the index for lowest-priced product. Hence, using Lemma 3, we can express the product 2's optimal price as follows:
\[
p^2_t = \frac{1}{2s} - \frac{1}{2s} p^1_t
\]
where \( t \in \{hi,lo\} \). Using the above relation, we can eliminate price variables for product 2 (i.e., \( p^2_t \)) in Model 9. Also, we use inventory balance equations to eliminate order quantities \( x^i_1 \). Next, we can aggregate the inventory variables. Let \( I \) be total inventory carried from high demand period to low demand period, i.e., \( I = I^1 + I^2 \). All these steps reduce the problem to one with 3 decision variables, i.e., high and low prices of product 1 and total inventory carry-over between high and low demand periods:
\[
\max_{2 \geq p^i_{hi} \geq p^1_{lo} \geq 0, t \in \{hi,lo\}} \{ f(p^i_{hi}, p^i_{lo}) - hI \text{ s. t. } d_{hi}(p^i_{hi}, p^1_{lo}) + d_{lo}(p^i_{hi}, p^1_{lo}) \leq 2q, d_{hi}(p^i_{hi}, p^1_{lo}) - I \leq q, p^2_t \geq p^1_t \}
\]
where \( d_{hi}(p^i_{hi}, p^1_{lo}) \) and \( d_{lo}(p^i_{hi}, p^1_{lo}) \) are total demands in high and low periods, respectively, i.e.,
\[
d_{hi}(p^i_{hi}, p^1_{lo}) = 2a - 2sp^i_{hi} \quad \text{and} \quad d_{lo}(p^i_{hi}, p^1_{lo}) = 2a + 2s (\alpha p^i_{hi} - (1 + \alpha)p^1_{lo})
\]
and \( f(p^i_{hi}, p^1_{lo}) \) is profit before inventory costs, i.e.,
\[
f(p^i_{hi}, p^1_{lo}) = \sum_{t \in \{hi,lo\}} (2(p^i_t - c)(a - sp^i_t)) + \sum_{t \in \{hi,lo\}} \frac{1}{4s} (a - sp^i_t)^2 + 2as(p^i_t - c)(p^i_{hi} - p^1_{lo})
\]
Note that \( p_t^2 \geq p_t^1 \) is automatically satisfied as long as \( 0 \leq p_t^1 \leq a/s \) since \( p_t^2 = \frac{1}{2} \tilde{p} + \frac{1}{2} p_t^1 \). Therefore, we can drop \( p_t^2 \geq p_t^1 \) from the formulation. Also in the analysis, we assume that \( 0 \leq p_t^1 \leq a/s \) and afterwards show that it is satisfied by the optimal prices. So we will solve the following problem:

\[
\max_{p_t^1, \geq n_t, t \geq 0} \left\{ f(p_t^1, p_t^1) - hi \text{ s.t. } d_{nt}(p_t^1, p_t^1) + d_{nt}(p_t^1, p_t^1) \leq 2q, d_{nt}(p_t^1, p_t^1) - I \leq q \right\}
\]  

(28)

Let \( \lambda_1 \geq 0 \) and \( \lambda_2 \geq 0 \) be the lagrangian variables for constraints 1 and 2, respectively. Let \( p = (p_t^1, p_t^1) \). Since Model 28 is a concave optimization problem with linear constraints, optimal solution \((p^*, I^*)\) has to satisfy the following KKT conditions: \( \nabla f(p) = \lambda_1 (\nabla d_{nt}(p) + \nabla d_{nt}(p)) + \lambda_2 (\nabla d_{nt}(p) - h + \lambda_2 - 0, I(-h + \lambda_2) - I \leq q, I_2(d_{nt}(p) - I - q) = 0, d_{nt}(p) + d_{nt}(p) - 2q, \lambda_1 (d_{nt}(p) + d_{nt}(p) - 2q) = 0, p_t^1 \geq p_t^1 \). Depending on whether \( I = 0 \) or \( I > 0, \lambda_1 = 0 \) or \( \lambda_1 > 0, \lambda_2 = 0 \) or \( \lambda_2 > 0 \), we have \( 2 \times 2 \times 2 = 8 \) cases:

- **Case-1: \((I = 0, \lambda_1 = 0, \lambda_2 = 0)\)**. Note that in this case, first KKT condition transform into \( \nabla f(p) = 0 \). The following prices satisfy this condition

\[
p_t^1 = \begin{cases} 
  a/s - (\tilde{p} - c) & t = hi \\
  a/s - (\tilde{p} - c) & t = lo 
\end{cases}
\]  

(29)

In order to meet capacity constraint in the second period, i.e., \( d_{nt}(p) \leq q, q \) must satisfy \( q \geq q_4 = 8(a - sc)^{7+11n+2n^2} \). Note that prices above satisfy \( p_t^1 \geq p_t^1 \). This implies that \( d_{nt}(p) \leq d_{nt}(p) \), hence \( d_{nt}(p) + d_{nt}(p) \leq 2q \) automatically holds. Therefore, the prices in Eqs (29) satisfy all KKT conditions, hence are optimal solution to Model 28 when \( q \geq q_4 \).

- **Case-2: \((I = 0, \lambda_1 > 0, \lambda_2 = 0)\)**. In this case, KKT conditions consist of \( \nabla f(p) = \lambda_1 (\nabla d_{nt}(p) + \nabla d_{nt}(p)) + d_{nt}(p) + d_{nt}(p) = 2q, d_{nt}(p) \leq q, p_t^1 \geq p_t^1 \). The prices that satisfy these equations are as follows:

\[
p_t^1 = \begin{cases} 
  a/s - \frac{7}{2} \frac{7+2+2n^2}{8 \alpha + 7n^2} & t = hi \\
  a/s - \frac{7}{2} \frac{7+2+2n^2}{8 \alpha + 7n^2} & t = lo 
\end{cases}
\]  

(30)

Note that \( p_t^1 \geq p_t^1 \), hence it implies that \( d_{nt}(p) \leq d_{nt}(p) \). Therefore, \( d_{nt}(p) \geq q \), which contradicts with KKT conditions. Hence, Case-2 does not lead to a feasible solution.

- **Case-3: \((I = 0, \lambda_1 = 0, \lambda_2 > 0)\)**. In this case, KKT conditions consist of \( \nabla f(p) = \lambda_1 (\nabla d_{nt}(p) + \nabla d_{nt}(p)) - h + \lambda_2 \leq 0, d_{nt}(p) = q, d_{nt}(p) + d_{nt}(p) \leq 2q, p_t^1 \geq p_t^1 \). The prices that satisfy these equations are as follows:

\[
p_t^1 = \begin{cases} 
  a/s - (\tilde{p} - c) + \frac{8}{2} \frac{8+2n^2}{8 \alpha + 8n^2} + q/s & t = hi \\
  a/s - (\tilde{p} - c) - q/s & t = lo 
\end{cases}
\]  

(31)

Three KKT inequalities (i.e., \( 0 \leq \lambda_2 \leq h, d_{nt}(p) + d_{nt}(p) \leq 2q \)) give rise to three conditions for \( q \geq q_3 = \frac{8(a - sc)(7+11n+2n^2)}{49+56n+16n^2} \), \( q \geq q = 8(a - sc)^{7+11n+2n^2} \), and \( q \leq q_4 = \frac{8(a - sc)(7+11n+2n^2)}{49+56n+16n^2} \). It is easy to check that all the KKT conditions are satisfied by these conditions. Hence, prices in Eqs (31) are optimal solution to Model 28 when \( q \geq q_3, q \geq q \) and \( q \leq q_4 \).

- **Case-4: \((I = 0, \lambda_1 > 0, \lambda_2 > 0)\)**. In this case, KKT conditions consist of \( \nabla f(p) = \lambda_1 (\nabla d_{nt}(p) + \nabla d_{nt}(p)) + \lambda_2 (\nabla d_{nt}(p) - h + \lambda_2 - 0, I(-h + \lambda_2) - I \leq q, d_{nt}(p) = q, d_{nt}(p) + d_{nt}(p) = 2q, p_t^1 \geq p_t^1 \). The prices that satisfy these equations are as follows:

\[
p_t^1 = \begin{cases} 
  a/s - q/(2s) & t = hi \\
  a/s - q/(2s) & t = lo 
\end{cases}
\]  

(32)

Note that \( p_t^1 = p_t^1 \). Three KKT conditions (i.e., \( \lambda_2 \geq 0, \lambda_2 \geq h \)) give rise to three conditions for \( q \geq 0, q \leq \tilde{q} = 8(a - sc)^{7+11n+2n^2} \), and \( q \leq q_1 = \frac{2n(a + sc)(1+a)}{4 \alpha} \). It is easy to check that all KKT conditions are satisfied by these conditions. Hence, prices in Eqs (32) are optimal solution to Model 28 when \( q \geq 0, q \leq \tilde{q} \) and \( q \leq q_1 \).
- **Case-5:** \( I > 0, \lambda_1 = 0, \lambda_2 = 0 \) KKT constraint, \(-h + \lambda_2 \leq 0\) can not hold when \( \lambda_2 = 0 \) since \( h \) is positive, therefore, this case leads to a contradiction.

- **Case-6:** \( I > 0, \lambda_1 > 0, \lambda_2 = 0 \) Due to the similar argument above, this case also leads to a contradiction.

- **Case-7:** \( I > 0, \lambda_1 = 0, \lambda_2 > 0 \) In this case, KKT conditions consist of \( \{ \nabla f(p) = \lambda_2 \nabla d_{lo}(p), -h + \lambda_2 = 0, d_{lo}(p) - I = q, h_{hi}(p) + d_{lo}(p) \leq 2q, p_{hi}^1 \geq p_{lo}^1 \}\). The prices that satisfy these equations are as follows:

\[
p_{hi}^1 = \begin{cases} \frac{a}{s} - \frac{q_1}{(2s)} \frac{7 + 3a}{2} - 2b_h \frac{1 + a}{2} & t = hi \\ \frac{a}{s} - \frac{q_1}{(2s)} \frac{7 + 11a}{2} + 2h \frac{1 + a}{2} & t = lo \end{cases}
\]  

(33)

Two KKT inequalities (i.e., \( d_{lo}(p) \geq q, h_{hi}(p) + d_{lo}(p) \geq 2q \)) give rise to two conditions for \( q \): \( q \geq q_2 = \frac{8(a - \alpha c)(7 + 3a - 2a^2 - 2b_h(a + \alpha c)(7 + 11a - 2a^2))}{49 + 56a - 16a^2} \), and \( q \leq q_3 = \frac{8(a - \alpha c)(7 + 11a - 2a^2 - 2b_h(a + \alpha c)(7 + 14a + 6a^2))}{49 + 56a - 16a^2} \). We can also express optimal inventory solution \( I \) by using \( I = d_{lo}(p_{hi}^1, p_{lo}^1) - q \). It is easy to check that under these conditions all KKT conditions are satisfied. Hence, prices in Eqs 33 and \( I = d_{lo}(p_{hi}^1, p_{lo}^1) - q \) are optimal solution to Model 28 when \( q \geq q_2 \), and \( q \leq q_3 \).

- **Case-8:** \( I > 0, \lambda_1 > 0, \lambda_2 > 0 \) In this case, KKT conditions consist of \( \{ \nabla f(p) = \lambda_1 \nabla d_{hi}(p) + \nabla d_{lo}(p), -h + \lambda_2 = 0, d_{lo}(p) - I = q, h_{hi}(p) + d_{lo}(p) = 2q, p_{hi}^1 \geq p_{lo}^1 \}\). The prices that satisfy these equations are as follows:

\[
p_{hi}^1 = \begin{cases} \frac{a}{s} - \frac{q_1}{(2s)} \frac{7 + 5a - 4a^2}{2} - 2b_h \frac{1 + a}{2} & t = hi \\ \frac{a}{s} - \frac{q_1}{(2s)} \frac{7 + 11a - 4a^2}{2} + 2h \frac{1 + a}{2} & t = lo \end{cases}
\]  

(34)

Two KKT conditions (i.e., \( \lambda_1 \geq 0, d_{lo}(p) \geq q \)) give rise to two conditions for \( q \): \( q \geq q_1 = \frac{2b_h(a + \alpha c)(1 + a)}{3a} \), and \( q \leq q_2 = \frac{8(a - \alpha c)(7 + 3a - 2a^2 - 2b_h(a + \alpha c)(7 + 11a - 2a^2))}{49 + 56a - 16a^2} \). Similarly, we can also express optimal inventory solution \( I \) by using \( I = d_{lo}(p_{hi}^1, p_{lo}^1) - q \). It is easy to check that under these conditions all KKT conditions are satisfied. Hence, prices in Eqs 34 and \( I = d_{lo}(p_{hi}^1, p_{lo}^1) - q \) are optimal solution to Model 28 when \( q \geq q_1 \), and \( q \leq q_2 \).

Finally, combining Equations 29, 31, 32, 33, and 34, and using \( p_{lo}^2 = \frac{a}{s} + \frac{1}{2} p_{hi}^1 \), we can obtain the complete characterization for optimal high and low prices for products 1 and 2 as summarized in Table 1.